Curvature bounds and heat kernels: discrete versus continuous spaces

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematish-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Anca-Iuliana Bonciocat
aus
Slatina, Rumänien

Bonn 2008

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Facultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter: Prof. Dr. Karl-Theodor Sturm
2. Gutachter: Prof. Dr. Lucian Beznea

Tag der Promotion: 12 Juli 2008

Abstract

We introduce and study rough (approximate) lower curvature bounds and rough curvature-dimension conditions for discrete spaces and for graphs. These notions extend the ones introduced in [St06a] and [St06b] to a larger class of non-geodesic metric measure spaces. They are stable under an appropriate notion of convergence in the sense that the metric measure space which is approximated by a sequence of discrete spaces with rough curvature $\geq K$ will have curvature $\geq K$ in the sense of [St06a]. Moreover, in the converse direction, discretizations of metric measure spaces with curvature $\geq K$ will have rough curvature $\geq K$. We apply our results to concrete examples of homogeneous planar graphs. We derive perturbed transportation cost inequalities, that imply mass concentration and exponential integrability of Lipschitz maps. For spaces that satisfy a rough curvature-dimension condition we prove a generalized Brunn-Minkowski inequality and a Bonnet-Myers type theorem.

Furthermore, we study Dirichlet forms on finite graphs and their approximations by Dirichlet forms on tubular neighborhoods. Our approach is based on a functional analytic concept of convergence of operators and quadratic forms with changing L_2-spaces, which uses the notion of measured Gromov-Hausdorff convergence for the underlying spaces. The convergence of the Dirichlet forms entails the convergence of the associated semigroups, resolvents and spectra to the corresponding objects on the graph.

Contents

Introduction

One of the challenging problems in mathematics – applied mathematics as well as pure mathematics – is to develop appropriate mathematical models for microstructures, as well as for discrete settings. Many very recent technical developments ask for new mathematical descriptions.

Discrete mathematics has become popular in the recent decades because of its applications to computer science. Triangulations of manifolds and discretizations of continuous spaces are very useful tools in digital geometry or computational geometry. The digital geometry deals with two main problems, inverse to each other: on one hand constructing digitized representations of objects, with a special emphasize on efficiency and precision, and on the other hand reconstructing "real" objects or their properties (length, area, volume, curvature, surface area) from digital images. Such a study requires of course a better understanding of the geometrical aspects of discrete spaces.

As a first step, geodesic metric spaces are very natural generalizations of manifolds. There are many recent developments in studying the geometry of such spaces. Even from the fifties a notion of lower curvature bounds for metric spaces was introduced by Alexandrov in [Al51], in terms of comparison properties for geodesic triangles. This notion gives the usual sectional curvature bounds when applied to Riemannian manifolds and it is stable under the Gromov-Hausdorff convergence, introduced in [Gro99].

More recently, a generalized notion of Ricci curvature bounds for metric measure spaces (M, d, m) was introduced and studied by K. T. Sturm in [St06a]; a closely related theory has been developed independently by J. Lott and C. Villani in [LV06]. The approach presented in [St06a] is based on convexity properties of the relative entropy $\mathrm{Ent}(\cdot|m)$ regarded as a function on the L_2-Wasserstein space of probability measures on the metric space (M, d). This lower curvature bound is stable under an appropriate notion of $\mathbb{D}$-convergence of metric measure spaces. The second paper [St06b] has treated the "finite dimensional" case, namely metric measure spaces that satisfy the so-called curvature-dimension condition $\mathsf{CD}(K, N)$, where K plays the role of the lower curvature bound and N the one of the upper dimension bound. The condition $\mathsf{CD}(K, N)$ represents the geometric counterpart of the analytic curvature-dimension condition introduced by D. Bakry and M. Émery in [BE85].

These generalizations required the Wasserstein space of probability measures (and thus in turn the underlying space) to be a geodesic space. Therefore, in the original form they will not apply to discrete spaces. Moreover, if we consider a graph, more precisely the union of the edges of a graph, as a metric space it will have no lower curvature bound in the sense of [St06a], since the vertices will be branch points of geodesics which destroy the K-convexity of the entropy.

Our point of view will come across coarse geometry, which studies the "large scale" properties of spaces (see for instance [Ro03] for an introduction). In various contexts, one notices that the relevant geometric properties of metric spaces are the coarse ones. A discrete space can get a geometric shape when we move the observation point far away from it; then all the original holes and gaps are not visible anymore and the space looks rather like a connected and continuous one. It is the point of view that led M. Gromov to his notion of hyperbolic group, which is a group "coarsely negatively curved" (in a certain combinatorial sense).

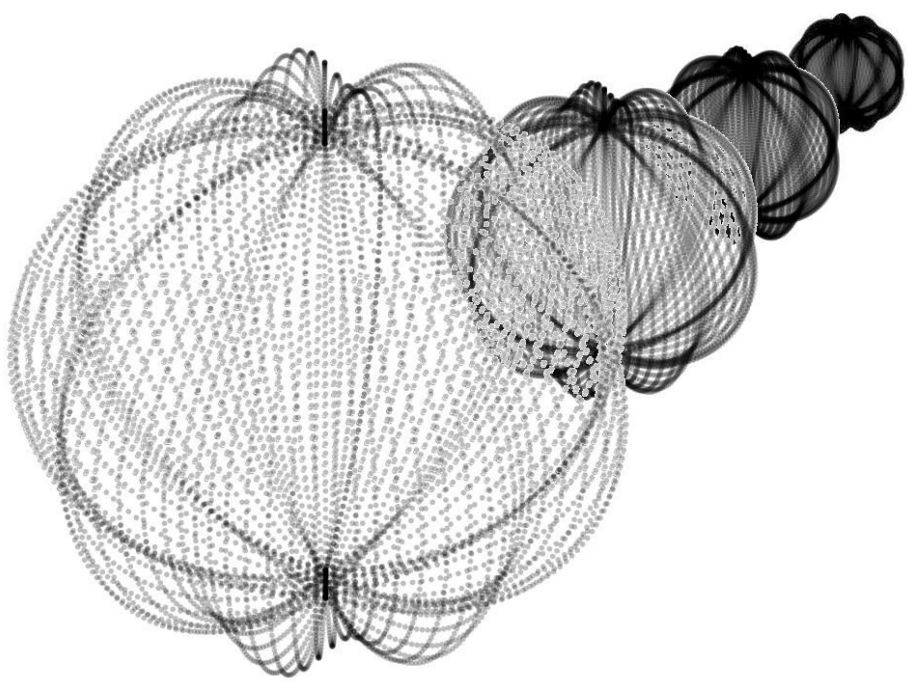

Figure 1

We develop a notion of rough curvature bounds for discrete spaces, as well as a rough curvature-dimension condition, based on the concept of optimal mass transportation. These rough curvature bounds will depend on a real parameter $h > 0$, which should be considered as a natural length scale of the underlying discrete space or as the scale on which we have to look at the space. For a metric graph, for instance, this parameter equals the maximal length of its edges (times some constant). The approach presented here will follow the one from [St06a] and it will be particularly concerned with removing the connectivity assumptions of the geodesic structure required there. This difficulty will be overcome in the following way: mass transportation and convexity properties of the relative entropy will be studied along h-geodesics. For instance instead of midpoints of a given pair of points x_0, x_1 we look at h-midpoints which are points y with $\mathsf{d}(x_0, y) \leq \frac{1}{2}\,\mathsf{d}(x_0, x_1) + h$ and $\mathsf{d}(x_1, y) \leq \frac{1}{2}\,\mathsf{d}(x_0, x_1) + h$.

In the first chapter we introduce and analyze rough curvature bounds for metric measure spaces, with emphasize on discrete spaces and graphs. Our first main result (Theorem 1.2.10) states that an arbitrary metric measure space (M, d, m) has curvature $\geq K$ (in the sense of [St06a]) provided it can be approximated by a sequence (M_h, d_h, m_h) of ("discrete") metric measure spaces with h-$\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \geq K_h$ with $K_h \to K$ as $h \to 0$. That is, this result allows to pass from discrete spaces to continuous limit spaces, reconstructing the curvature bound of the continuous space from the coarse curvature bounds of the approximating (possibly discrete) spaces.

The second main result (Theorem 1.3.1) states that the curvature bound will also be preserved under the converse procedure: Given any metric space (M, d, m) with curvature $\geq K$ and any $h > 0$ we define standard discretizations (M_h, d, m_h) of (M, d, m) with $\mathbb{D}((M_h, \mathsf{d}, m_h), (M, \mathsf{d}, m)) \to 0$ as $h \to 0$ and with the rough curvature bound h-$\mathbb{C}\mathrm{urv}(M_h, \mathsf{d}_h, m_h) \geq K$.

The stability under discretizations provides a series of concrete examples. We prove (Theorem 1.4.3) that every homogeneous planar graph has h-curvature $\geq K$ where K is given in terms of the degree, the dual degree and the edge length. To be more precise, both the set $M = V$ of vertices, equipped with the counting measure, as well as the union $M = \bigcup_{e \in E} e$ of edges equipped with one-dimensional Lebesgue measure will be metric measure spaces with h-curvature $\geq K$, where the metric is the one induced by the Riemannian distance of the 2-dimensional Riemannian manifold whose discretization will be our given graph. Our notion of h-curvature yields the precise value for K if we consider discretizations of hyperbolic spaces. It is also related to some notions of combinatorial curvature, see e.g. [Gro87], [Is90], [Hi01], [Fo03].

In section 1.5 we show that positive rough curvature bound implies a perturbed transportation cost inequality, weaker than what is usually called the Talagrand inequality. However, it still implies concentration of the reference measure m and exponential integrability of the Lipschitz functions with respect to m.

The second chapter introduces the rough curvature-dimension condition h-$\mathsf{CD}(K, N)$ for metric measure spaces, coming with an additional upper bound for the "dimension". The planar graphs for instance are discrete analogues of connected Riemannian surfaces, therefore they deserve to be considered 2-dimensional discrete spaces. Besides, an upper bound for the dimension would be expected to bring, by analogy with the finite dimensional Riemannian manifolds, more geometrical consequences in our discrete setting.

In section 2.2 we define the rough curvature-dimension condition and give some basic properties. We show that the rough curvature bound presented in Chapter 1 can be seen as a limit case or as an h-$\mathsf{CD}(K, \infty)$ rough curvature-dimension condition.

Section 2.3 provides some geometrical consequences of the rough curvature-dimension condition. We prove a generalized Brunn-Minkowski inequality that holds under an h-$\mathsf{CD}(K, N)$ property. Furthermore, we give a Bonnet-Myers type theorem, which states that a metric measure space that satisfies an h-$\mathsf{CD}(K, N)$ condition with $K > 0$ has bounded diameter. Consequently, planar graphs that fulfill an h-$\mathsf{CD}(K, N)$ condition with $K > 0$ must be finite.

The stability issue under $\mathbb{D}$-convergence is treated within section 2.4. Theorem 2.4.1 states that any (continuous) metric measure space that can be approximated in the metric $\mathbb{D}$ by a family of (possibly discrete) metric measure spaces (M_h, d_h, m_h) with bounded diameter L_h, with a rough curvature-dimension condition h-$\mathsf{CD}(K_h, N_h)$ satisfied and with $(K_h, N_h, L_h) \to (K, L, N)$, will satisfy a curvature-dimension condition $\mathsf{CD}(K, N)$ and will have diameter $\leq L$.

In section 2.5 we show that our curvature-dimension condition will be preserved trough the converse procedure, by discretizing a continuous space that fulfills it. Theorem 2.5.1 shows that whenever we consider a discretization (M_h, d, m_h) with sufficiently small mash size of a space (M, d, m) that satisfies some $\mathsf{CD}(K, N)$ condition, the discretization will satisfy the h-$\mathsf{CD}(K, N)$ property.

If discrete spaces can be seen as almost continuous and solid from a remote observation point, some smooth and quite consistent objects might look like arousing singularities, if seen from a distance. A net of pipes crossing each other would actually look like a graph, the smooth picture would be like shrunk towards its skeleton. One can expect that some properties of the approximating smooth object will be carried further to the singular limit, but experience shows that others degenerate, surprisingly sometimes. In this case one should rather give up the "large scale" point of view in the favor of a closer look around the region that will give rise to the singularity. Metric graphs are used to model various real graph-like structures, whose transverse size is small but not zero, and it is important to know how such thin systems approximate an ideal graph when their width goes to zero.

Convergence of Riemannian manifolds, or more generally convergence of metric spaces is a well established concept in geometry [Gro99]. The situation becomes more complicated if the focus lies not only on the convergence of spaces but also on convergence of semigroups, generators, spectra etc. Our aim is to study the convergence behavior of Laplace operators and heat kernels on tubular neighborhoods of graphs towards the "canonical Laplacian" on the graph itself.

A consistent progress has been done in the recent years in classifying possible boundary conditions for Laplace operators on graphs (see [ES89], [KoS99]). The convergence of the spectra of the Laplacian on the "graph-like" approximating manifolds towards the spectrum of the Laplacian on the graph gives important informations for physicists (see [Ku02]). The case of Neumann boundary condition has been solved for various approximating families of manifolds, the limit depending on the rate of convergence of the volume of the vertex-neighborhoods with respect to the volume of the edge-neighborhoods (see [EP05]). It has been shown that the kth eigenvalue of the Neumann Laplacian on the manifold converges to the kth eigenvalue of the corresponding operator on the graph, and the standard boundary condition on the graph is basically the so-called Kirchhoff boundary condition. Mixed boundary conditions for the approximating sequence of manifolds have been considered in [Po05], [Gr07]. In [Po05] for instance the main result states the convergence of the spectra of a family of approximating open sets from $\mathbb{R}^2$ with small vertex neighborhoods and with a mixed boundary condition towards the spectrum of the Laplacian on the graph with Dirichlet boundary condition, which is actually a graph operator without coupling between edges. The paper [Po06] studies the approximations with non-compact manifolds and in the Neumann case gives, besides the convergence of spectra, the norm convergence of resolvents.

Sidova, Smolyanov, v.Weizsäcker and Wittich ([SW04a] and [SW04b]) have studied Brownian motions on tubular neighborhoods of embedded manifolds and their convergence behavior, finding corrections terms related to the curvature of the embedding.

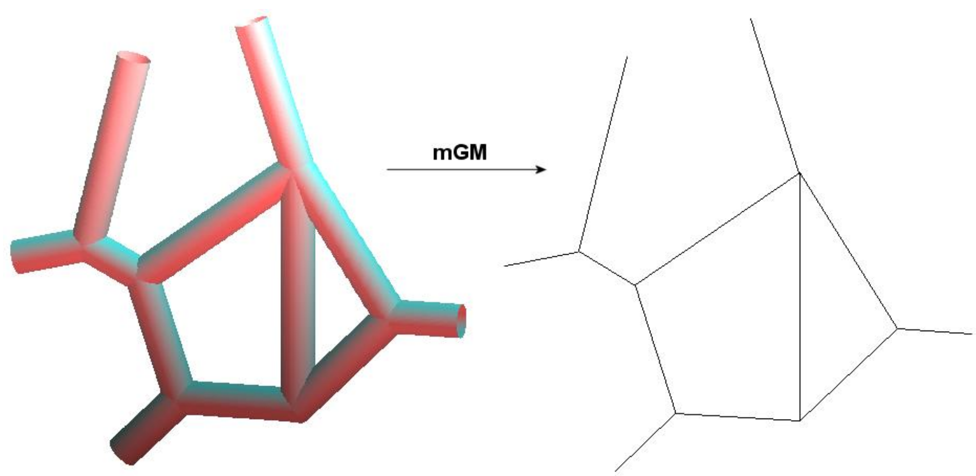

Figure 2

In Chapter 3 we shall study the convergence of the Laplace operators on tubular neighborhoods of N-spiders as an application of a functional analytic theory on changing L^2-spaces. In their recent work [KS03] K. Kuwae and T. Shioya studied convergence of operators and quadratic forms which are not necessarily defined on the same Hilbert space. Another definition of convergence on changing Hilbert spaces for the case of the Hilbert spaces of the type $H_n = L^2(\sigma_n\, dx)$, with $\sigma_n\, dx \to \sigma\, dx$ vaguely on $\mathbf{R}^d$, was introduced by V. Zhikov in [Zhi98]. Although the two definitions of convergence differ, it is shown in the paper of A. Kolesnikov [Kol05] that the two approaches are equivalent.

The definition of convergence given in [KS03] uses the notion of Gromov-Hausdorff convergence for the underlying spaces. The Gromov-Hausdorff convergence of metric spaces has been introduced by M. Gromov in [Gro99], and further the measured Gromov-Hausdorff convergence by Fukaya [Fu87]. K. Kuwae and T. Shioya have developed (extending the Γ-convergence and Mosco-convergence) a general theory of convergence of spectral structures, which defines the convergence of the whole machinery semigroups-resolvents-spectra-Dirichlet forms from the approximating sequence to the limit space. The basic definitions and the main results from [KS03] are recalled within section 3.1.

In section 3.2 we consider various tubular open (bounded) domains that can approximate an edge, as basic building blocks for constructing further graph-like neighborhoods. Beside the cylindrical neighborhood we study weighted tubes with variable width that satisfy some smoothness assumptions on the boundary. We prove the convergence of the spectral structure from the tube towards the spectral structure on the edge.

Section 3.3 deals with the case of a graph M with a single branch point and a finite number of weighted edges, the so-called "spider", and the canonical Laplacian on the graph with Kirchhoff boundary condition in the branch point. We consider approximations with open bounded tubular domains, consisting of cylindrical edge-neighborhoods and vertex-neighborhoods whose decay rate in terms of volume is prescribed between given values that correspond to a faster decay of the vertex-neighborhoods in comparison with the one of the edge-neighborhoods.

Denoting the edges of the spider by $e_1, e_2, \ldots, e_N$ we define the Dirichlet form on the graph as

$$\mathcal{E}(u) := \sum_{i=1}^{N} \int_{e_i} |u'(x)|^2 \rho_i(x) dx$$

having the domain $\mathcal{D}(\mathcal{E}) = \{u \in C(M) : u|_{e_i} \in H^1(e_i),\ i = 1, 2, ..., N\}$ and the classical Dirichlet forms on the tubes

$$\mathcal{E}^n(u_n) := \int_{M_n} |\nabla u_n|^2\ dm_n, \qquad u_n \in H^1(M_n),\ n \in \mathbf{N}.$$

Under mild assumption on the measure m_n and certain smoothness assumptions on the underlying domains, our main results are:

1) The asymptotic compactness of the sequence $\{\mathcal{E}^n\}_n$, namely for any sequence $\{u_n\}_n$ with $u_n \in L^2(M_n, m_n)$ and $\limsup_n(\mathcal{E}_n(u_n) + \|u_n\|^2_{L^2(M_n,m_n)}) < \infty$, there exists a strongly convergent subsequence $\{u_n\}_n$.

2) $\{\mathcal{E}^n\}_n$ is Γ-convergent to $\mathcal{E}$.

According to [KS03], these results imply also the convergence of the associated resolvents, semigroups and spectra.

The N-spider-like neighborhoods, combined with the more general edge-neighborhoods studied in section 3.2, give various graph-like neighborhoods for which the convergence of the whole spectral structure holds.

Chapter 1

Rough curvature bounds for metric measure spaces

We develop a notion of rough curvature bounds for discrete spaces, based on the concept of optimal mass transportation. These rough curvature bounds will depend on a real parameter $h > 0$, which should be considered as a natural length scale of the underlying discrete space or as the scale on which we have to look at the space. For a metric graph, for instance, this parameter equals the maximal length of its edges (times some constant).

The approach presented here will follow the one from [St06a], where a notion of lower curvature bounds for metric measure spaces has been introduced. That notion required the Wasserstein space of probability measures (and thus in turn the underlying space) to be a geodesic space. Therefore, in the original form it cannot apply to discrete spaces. Besides, metric graphs will have no lower curvature bound in the sense of [St06a], since the vertices will be branch points of geodesics which destroy the K-convexity of the entropy. The modification to be presented here overcomes this difficulty in the following way: mass transportation and convexity properties of the relative entropy will be studied along h-geodesics instead of geodesics.

In the first section we give an overview of the material already existing in the literature, particularly the notion of lower curvature bound for (continuous) metric measure spaces.

The two main results we prove within this chapter are in some sense inverse to each other: on one hand reconstructing the curvature bound of a continuous space from the rough curvature bounds of approximating discrete spaces with mesh size tending to zero (Theorem 1.2.10), and on another hand the deduction of the rough curvature bounds of dicretizations of a continuous space from the curvature bound of the latter (Theorem 1.3.1).

In section 1.4 we apply our results to concrete examples. We prove (Theorem 1.4.3) that every homogeneous planar graph has h-curvature $\geq K$ where K is given in terms of the degree, the dual degree and the edge length.

In the final section we show that positive rough curvature bound implies a perturbed transportation cost inequality, weaker than what is usually called the Talagrand inequality. However, it still implies concentration of the reference measure m and exponential integrability of the Lipschitz functions with respect to m.

1.1 Preliminaries

Throughout this chapter, a *metric measure space* will always be a triple (M, d, m) where (M, d) is a complete separable metric space and m is a measure on M (equipped with its Borel σ-algebra $\mathcal{B}(M)$) which is locally finite in the sense that $m(B_r(x)) < \infty$ for all $x \in M$ and all sufficiently small $r > 0$. We say that the metric measure space (M, d, m) is *normalized* if $m(M) = 1$.

Two metric measure spaces (M, d, m) and (M', d', m') are called *isomorphic* iff there exists an isometry $\psi : M_0 \to M_0'$ between the supports $M_0 := \text{supp}[m] \subset M$ and $M_0' := \text{supp}[m'] \subset M'$ such that $\psi_* m = m'$. The *diameter* of a metric measure space (M, d, m) will be the diameter of the metric space $(\text{supp}[m], d)$.

We shall use the notion of L_2-*transportation distance* $\mathbb{D}$ for two metric measure spaces (M, d, m) and (M', d', m'), as defined in [St06a]:

$$\mathbb{D}((M, \mathsf{d}, m), (M', \mathsf{d}', m')) = \inf \left(\int_{M \sqcup M'} \hat{\mathsf{d}}^2(x, y) dq(x, y) \right)^{1/2},$$

where $\hat{\mathsf{d}}$ ranges over all couplings of d and d' and q ranges over all couplings of m and m'. Here a measure q on the product space $M \times M'$ is a *coupling of m and m'* if $q(A \times M') = m(A)$ and $q(M \times A') = m'(A')$ for all measurable $A \subset M, A' \subset M'$; a pseudo-metric $\hat{\mathsf{d}}$ on the disjoint union $M \sqcup M'$ is a *coupling of* d *and* d' if $\hat{\mathsf{d}}(x, y) = \mathsf{d}(x, y)$ and $\hat{\mathsf{d}}(x', y') = \mathsf{d}'(x', y')$ for all $x, y \in \text{supp}[m] \subset M$ and all $x', y' \in \text{supp}[m'] \subset M'$.

The L_2-transportation distance $\mathbb{D}$ defines a complete separable length metric on the family of all isomorphism classes of normalized metric measure spaces (M, d, m) for which $\int_M \mathsf{d}^2(z, x) dm(x) < \infty$ for some (hence all) $z \in M$. The notion of $\mathbb{D}$-convergence is closely related to the one of measured Gromov-Hausdorff convergence introduced in [Fu87].

Recall that a sequence of compact and normalized metric measure spaces $\{(M_n, \mathsf{d}_n, m_n)\}_{n \in \mathbb{N}}$ converges in the sense of *measured Gromov-Hausdorff convergence* (briefly, mGH-converges) to a compact normalized metric measure space (M, d, m) iff there exist a sequence of numbers $\epsilon_n \searrow 0$ and a sequence of measurable maps $f_n : M_n \to M$ such that for all $x, y \in M_n$, $|\,\mathsf{d}(f_n(x), f_n(y)) - \mathsf{d}_n(x, y)| \leq \epsilon_n$, for any $x \in M$ there exists $y \in M_n$ with $\mathsf{d}(f_n(y), x) \leq \epsilon_n$ and such that $(f_n)_* m_n \to m$

weakly on M for $n \to \infty$. According to Lemma 3.17 in [St06a], any mGH-convergent sequence of normalized metric measure spaces is also $\mathbb{D}$-convergent; for any sequence of normalized compact metric measure spaces with full supports and with uniform bounds for the doubling constants and for the diameters the notion of mGH-convergence is equivalent to the one of $\mathbb{D}$-convergence.

It is easy to see that $\mathbb{D}((M, \mathsf{d}, m), (M', \mathsf{d}', m')) = \inf \hat{\mathsf{d}}_W(\psi_* m, \psi'_* m')$ where the inf is taken over all metric spaces $(\hat{M}, \hat{\mathsf{d}})$ with isometric embeddings $\psi : M_0 \hookrightarrow \hat{M}$, $\psi' : M'_0 \hookrightarrow \hat{M}$ of the supports M_0 and M'_0 of m and m', respectively, and where $\hat{\mathsf{d}}_W$ denotes the L_2-Wasserstein distance derived from the metric $\hat{\mathsf{d}}$. Recall that for any metric space (M, d) the L_2-*Wasserstein distance* between two measures μ and ν on M is defined as

$$\mathsf{d}_W(\mu,\nu) = \inf \left\{ \left(\int_{M\times M} \mathsf{d}^2(x,y)dq(x,y) \right)^{1/2} : q \text{ is a coupling of } \mu \text{ and } \nu \right\},$$

with the convention $\inf \emptyset = \infty$. For further details about the Wasserstein distance see the monograph [Vi03]. We denote by $\mathcal{P}_2(M, \mathsf{d})$ the space of all probability measures ν which have finite second moments $\int_M \mathsf{d}^2(o,x)d\nu(x) < \infty$ for some (hence all) $o \in M$.

For a given metric measure space (M, d, m) we put $\mathcal{P}_2(M, \mathsf{d}, m)$ the space of all probability measures $\nu \in \mathcal{P}_2(M, \mathsf{d})$ which are absolutely continuous w.r.t. m. If $\nu = \rho \cdot m \in \mathcal{P}_2(M, \mathsf{d}, m)$ we consider the *relative entropy* of ν with respect to m defined by $\mathrm{Ent}(\nu|m) := \lim_{\epsilon \searrow 0} \int_{\{\rho > \epsilon\}} \rho \log \rho \, dm$. We denote by $\mathcal{P}^*_2(M, \mathsf{d}, m)$ the subspace of measures $\nu \in \mathcal{P}_2(M, \mathsf{d}, m)$ of finite entropy $\mathrm{Ent}(\nu|m) < \infty$.

In classical Riemannian geometry, given a point x in a Riemannian manifold the Ricci curvature Ric_x is defined on the tangent space T_xM as

$$\mathrm{Ric}_x(v,v) := \mathrm{trace}\{w \to \mathcal{R}(v,w)v\}, \ v \in T_xM,$$

where $\mathcal{R}$ is the curvature tensor. The Ricci curvature Ric_x measures the non-euclidian behavior of the manifold in direction v.

In the paper [RS05] the authors give the following characterization of the Ricci curvature bound for Riemannian manifolds.

Theorem 1.1.1. *For any smooth connected Riemannian manifold M with intrinsic metric d and volume measure m and any $K \in \mathbb{R}$ the following properties are equivalent:*

(i) $\mathrm{Ric}_x(v,v) \geq K|v|^2$ *for $x \in M$ and $v \in T_x(M)$.*

(ii) The entropy $Ent(\cdot|m)$ is displacement K-convex on $\mathcal{P}_2(M)$ in the sense that for each geodesic $\gamma : [0,1] \to \mathcal{P}_2(M)$ and for each $t \in [0,1]$

$$\mathrm{Ent}(\gamma(t)|m) \leq (1-t)\mathrm{Ent}(\gamma(0)|m) + t\mathrm{Ent}(\gamma(1)|m) - \frac{K}{2} t(1-t) \, \mathsf{d}^2_W(\gamma(0),\gamma(1)).$$

This characterization makes no use of the differentiability issue and condition (ii) can be posed in any (geodesic) metric measure spaces. Therefore (ii) might stand as a definition for a lower Ricci curvature bound. Indeed [St06a] proves stability under $\mathbb{D}$-convergence and pinpoints a series of results that correspond to classical theorems from Riemannian geometry involving Ricci curvature.

We recall here the definitions of the lower curvature bounds for metric measure spaces introduced in [St06a]:

Definition 1.1.2. *(i) A metric measure space (M, d, m) has* curvature $\geq K$ *for some number $K \in \mathbb{R}$ iff the relative entropy $\mathrm{Ent}(\cdot|m)$ is weakly K-convex on $\mathcal{P}_2^*(M, \mathsf{d}, m)$ in the sense that for each pair $\nu_0, \nu_1 \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ there exists a geodesic $\Gamma : [0, 1] \to \mathcal{P}_2^*(M, \mathsf{d}, m)$ connecting ν_0 and ν_1 with*

$$\mathrm{Ent}(\Gamma(t)|m) \leq (1-t)\mathrm{Ent}(\Gamma(0)|m) + t\mathrm{Ent}(\Gamma(1)|m) - \frac{K}{2}t(1-t)\,\mathsf{d}_W^2(\Gamma(0), \Gamma(1)) \tag{1.1.1}$$

for all $t \in [0, 1]$.

(ii) The metric measure space (M, d, m) has curvature $\geq K$ in the lax sense *iff for each $\epsilon > 0$ and for each pair $\nu_0, \nu_1 \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ there exists an ϵ-midpoint $\eta \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ of ν_0 and ν_1 with*

$$\mathrm{Ent}(\eta|m) \leq \frac{1}{2}\mathrm{Ent}(\nu_0|m) + \frac{1}{2}\mathrm{Ent}(\nu_1|m) - \frac{K}{8}\,\mathsf{d}_W^2(\nu_0, \nu_1) + \epsilon. \tag{1.1.2}$$

Briefly, we shall write $\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \geq K$, respectively $\mathbb{C}\mathrm{urv}_{lax}(M, \mathsf{d}, m) \geq K$.

Recall that in a given metric space (M, d) a point y is an *ϵ-midpoint* of x_0 and x_1 if $\mathsf{d}(x_i, y) \leq \frac{1}{2}\,\mathsf{d}(x_0, x_1) + \epsilon$ for each $i = 0, 1$. We call y *midpoint* of x_0 and x_1 if $\mathsf{d}(x_i, y) \leq \frac{1}{2}\,\mathsf{d}(x_0, x_1)$ for $i = 0, 1$.

1.2 Rough curvature bounds for metric measure spaces

In order to adapt the notion of curvature bound to other spaces then geodesic spaces without branching we shall refer in this paper to a larger class of metric spaces:

Definition 1.2.1. Let $h > 0$ be given. We say that a metric space (M, d) is *h-rough geodesic* iff for each pair of points $x_0, x_1 \in M$ and each $t \in [0, 1]$ there exists a point $x_t \in M$ satisfying

$$\mathsf{d}(x_0, x_t) \leq t\,\mathsf{d}(x_0, x_1) + h, \;\; \mathsf{d}(x_t, x_1) \leq (1-t)\,\mathsf{d}(x_0, x_1) + h. \tag{1.2.1}$$

The point x_t will be referred to as the h-rough t-intermediate point between x_0 and x_1. The h-rough 1/2-intermediate point is actually the h-midpoint of x_0 and x_1.

Example 1.2.2. (i) Any nonempty set X with the discrete metric $\mathsf{d}(x,y) = 0$ for $x = y$ and 1 for $x \neq y$ is h-rough geodesic for any $h \geq 1/2$. In this case, any point is an h-midpoint of any pair of distinct points.

(ii) If $\epsilon > 0$ then the space $(\mathbb{R}^n, \mathsf{d})$ with the metric $\mathsf{d}(x,y) = |x-y| \wedge \epsilon$ is h-rough geodesic for $h \geq \epsilon/2$ (here $|\cdot|$ is the euclidian metric).

(iii) For $\epsilon > 0$ the space $(\mathbb{R}^n, \mathsf{d})$ with the metric $\mathsf{d}(x,y) = \sqrt{\epsilon|x-y| + |x-y|^2}$ is h-rough geodesic for each $h \geq \epsilon/4$.

The above examples are somewhat pathological. We actually have in mind the more friendly examples of discrete spaces and some geodesic spaces with branch points, e.g. graphs, that do not have curvature bounds as defined in [St06a].

For a discrete h-rough geodesic metric space (M, d) one should think of h as a discretization size or "resolution" of M. In an h-geodesic space a pair of points x and y is not necessarily connected by a geodesic but by a chain of points $x = x_0, x_1, \cdots, x_n = y$ having intermediate distance less then $h/2$.

In the sequel we will use two types of perturbations of the Wasserstein distance, defined as follows:

Definition 1.2.3. Let (M, d) be a metric space. For each $h > 0$ and any pair of measures $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d})$ put

$$\mathsf{d}_W^{\pm h}(\nu_0,\nu_1) := \inf\left\{\left(\int [(\mathsf{d}(x_0,x_1) \mp h)_+]^2\, dq(x_0,x_1)\right)^{1/2}\right\}, \tag{1.2.2}$$

where q ranges over all couplings of ν_0 and ν_1 and $(\cdot)_+$ denotes the positive part.

Remark 1.2.4. According to Theorem 4.1 from [Vi08] there exists a coupling for which the infimum in (1.2.2) is attaint. We will call it $+h$-optimal coupling (resp. $-h$-optimal coupling) of ν_0 and ν_1.

The two perturbations d_W^{+h} and d_W^{-h} are related to the Wasserstein distance d_W in the following way:

Lemma 1.2.5. *For any $h > 0$ we have*

(i) $\mathsf{d}_W^{+h} \leq \mathsf{d}_W \leq \mathsf{d}_W^{+h} + h$;

(ii) $\mathsf{d}_W \leq \mathsf{d}_W^{-h} \leq \mathsf{d}_W + h$.

Proof. (i) Let ν_0 and ν_1 be two probabilities in (M, d) and consider q an optimal coupling and q_{+h} a $+h$-optimal coupling of them. Then

$$\begin{aligned}
\mathsf{d}_W^{+h}(\nu_0,\nu_1) &= \left(\int \left[(\mathsf{d}(x_0,x_1)-h)_+\right]^2 dq_{+h}(x_0,x_1)\right)^{1/2} \\
&\le \left(\int \left[(\mathsf{d}(x_0,x_1)-h)_+\right]^2 dq(x_0,x_1)\right)^{1/2} \\
&\le \left(\int \mathsf{d}(x_0,x_1)^2 dq(x_0,x_1)\right)^{1/2} = \mathsf{d}_W(\nu_0,\nu_1)
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{d}_W(\nu_0,\nu_1) &= \left(\int \mathsf{d}(x_0,x_1)^2 dq(x_0,x_1)\right)^{1/2} \le \left(\int \mathsf{d}(x_0,x_1)^2 dq_{+h}(x_0,x_1)\right)^{1/2} \\
&\le \left(\int \left[(\mathsf{d}(x_0,x_1)-h)_+ + h\right]^2 dq_{+h}(x_0,x_1)\right)^{1/2} \le \mathsf{d}_W^{+h}(\nu_0,\nu_1)+h.
\end{aligned}$$

(ii) Similar to (i). □

With an elementary proof we have also a monotonicity property of $\mathsf{d}_W^{\pm h}$ in h:

Lemma 1.2.6. *Let $0 < h_1 < h_2$ be arbitrarily given. Then for each pair of probabilities ν_0 and ν_1*

(i) $\mathsf{d}_W^{-h_1}(\nu_0,\nu_1) < \mathsf{d}_W^{-h_2}(\nu_0,\nu_1)$;

(ii) $\mathsf{d}_W^{+h_1}(\nu_0,\nu_1) \ge \mathsf{d}_W^{+h_2}(\nu_0,\nu_1)$ *with strict inequality if and only if* $\mathsf{d}_W^{+h_1}(\nu_0,\nu_1) > 0$.

We introduce now the notion of rough lower curvature bound:

Definition 1.2.7. We say that a metric measure space (M, d, m) has *h-rough curvature* $\ge K$ for some numbers $h > 0$ and $K \in \mathbb{R}$ iff for each pair $\nu_0, \nu_1 \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ and for any $t \in [0,1]$ there exists an h-rough t-intermediate point $\eta_t \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ between ν_0 and ν_1 satisfying

$$\mathrm{Ent}(\eta_t|m) \le (1-t)\mathrm{Ent}(\nu_0|m) + t\mathrm{Ent}(\nu_1|m) - \frac{K}{2}t(1-t)\,\mathsf{d}_W^{\pm h}(\nu_0,\nu_1)^2, \qquad (1.2.3)$$

where the sign in $\mathsf{d}_W^{\pm h}(\nu_0,\nu_1)$ is chosen '+' if $K > 0$ and '−' if $K < 0$. Briefly, we write in this case h-$\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \ge K$.

Remark 1.2.8. We could also choose two parameters in the above definition, h for the intermediate midpoint and ϵ for the inequality (1.2.3). Having two parameters instead of one is not essentially useful for further results. One can always think of $h \vee \epsilon$ in the definition of rough curvature bound, which is an approximate notion.

Remark 1.2.9. (i) If (M, d, m) and (M', d', m') are two isomorphic metric measure spaces and $K \in \mathbb{R}$ then $h\text{-}\,\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \geq K$ if and only if $h\text{-}\,\mathbb{C}\mathrm{urv}(M', \mathsf{d}', m') \geq K$.

(ii) If (M, d, m) is a metric measure space and $\alpha, \beta > 0$ then $h\text{-}\,\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \geq K$ if and only if $\alpha h\text{-}\,\mathbb{C}\mathrm{urv}(M, \alpha\,\mathsf{d}, \beta m) \geq \frac{K}{\alpha^2}$, because $\mathrm{Ent}(\nu|\beta m) = \mathrm{Ent}(\nu|m) - \log \beta$, $(\alpha \cdot \mathsf{d})_W^{\pm h}(\nu_0, \nu_1) = \alpha \cdot \mathsf{d}_W^{\pm h}(\nu_0, \nu_1)$ and for $t \in [0,1]$ η_t is h-rough t-intermediate point between μ, ν with respect to d_W if and only if η_t is αh-rough t-intermediate point between μ, ν with respect to $(\alpha\,\mathsf{d})_W$.

Theorem 1.2.10. *Let (M, d, m) be a normalized metric measure space and consider $\{(M_h, \mathsf{d}_h, m_h)\}_{h>0}$ a family of normalized metric measure spaces with uniformly bounded diameter and with $h\text{-}\,\mathbb{C}\mathrm{urv}(M_h, \mathsf{d}_h, m_h) \geq K_h$ for $K_h \to K$ as $h \to 0$. If*

$$(M_h, \mathsf{d}_h, m_h) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$$

as $h \to 0$ then

$$\mathbb{C}\mathrm{urv}_{lax}(M, \mathsf{d}, m) \geq K.$$

If in addition M is compact then

$$\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \geq K.$$

Proof. Let $\{(M_h, \mathsf{d}_h, m_h)\}_{h>0}$ be a family of normalized discrete metric measure spaces. Assume that we have $\sup_{h>0} \mathrm{diam}(M_h, \mathsf{d}_h, m_h)$, $\mathrm{diam}(M, \mathsf{d}, m) \leq \Delta$ and $(M_h, \mathsf{d}_h, m_h) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$ as $h \to 0$. Now let $\epsilon > 0$ and $\nu_0 = \rho_0 m$, $\nu_1 = \rho_1 m \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ be given. Choose R with

$$\sup_{i=0,1} \mathrm{Ent}(\nu_i|m) + \frac{|K|}{8}\Delta^2 + \frac{\epsilon}{8}[\Delta^2 + 3|K|(2\Delta + 3\epsilon)] \leq R. \tag{1.2.4}$$

We have to deduce the existence of an ϵ-midpoint η which satisfies inequality (1.1.2). Choose $0 < h < \epsilon$ with $|K_h - K| < \epsilon$ and

$$\mathbb{D}((M_h, \mathsf{d}_h, m_h), (M, \mathsf{d}, m)) \leq \exp\left(-\frac{2 + 4\Delta^2 R}{\epsilon^2}\right). \tag{1.2.5}$$

One can define the canonical maps $Q'_h : \mathcal{P}_2(M, \mathsf{d}, m) \to \mathcal{P}_2(M_h, \mathsf{d}_h, m_h)$, $Q_h : \mathcal{P}_2(M_h, \mathsf{d}_h, m_h) \to \mathcal{P}_2(M, \mathsf{d}, m)$ like in the subsection 4.5 in [St06a]:

We consider q_h a coupling of m and m_h and $\hat{\mathsf{d}}_h$ a coupling of d and d_h such that

$$\int \hat{\mathsf{d}}_h^2(x,y)\,dq_h(x,y) \le 2\mathbb{D}^2((M,\,\mathsf{d},m),(M_h,\,\mathsf{d}_h,m_h)).$$

Let Q'_h and Q_h be the disintegrations of q_h w.r.t. m_h and m, resp., that is $dq_h(x,y) = Q'_h(y,dx)dm_h(y) = Q_h(x,dy)dm(x)$ and let $\hat{\Delta}$ denote the m-essential supremum of the map

$$x \mapsto \left[\int_{M_h} \hat{\mathsf{d}}_h^2(x,y)\,Q_h(x,dy)\right]^{1/2}.$$

In our case $\hat{\Delta} \le 2\Delta$.

For $\nu = \rho m \in \mathcal{P}_2(M,\,\mathsf{d},m)$ define $Q'_h(\nu) \in \mathcal{P}_2(M_h,\,\mathsf{d}_h,m_h)$ by $Q'_h(\nu) := \rho_h m_h$ where

$$\rho_h(y) := \int_M \rho(x)Q'_h(y,dx).$$

The map Q_h is defined similarly. Lemma 4.19 from [St06a] gives the following estimates:

$$\mathrm{Ent}(Q'_h(\nu)|m_h) \le \mathrm{Ent}(\nu|m) \text{ for all } \nu = \rho\, m \tag{1.2.6}$$

and

$$\mathsf{d}_W^2(\nu, Q'_h(\nu)) \le \frac{2 + \hat{\Delta}^2 \cdot \mathrm{Ent}(\nu|m)}{-\log \mathbb{D}((M,\,\mathsf{d},m),(M_h,\,\mathsf{d}_h,m_h))}. \tag{1.2.7}$$

provided $\mathbb{D}((M,\,\mathsf{d},m),(M_h,\,\mathsf{d}_h,m_h)) < 1$. Analogous estimates hold for Q_h.

For our given $\nu_0 = \rho_0 m$, $\nu_1 = \rho_1 m \in \mathcal{P}_2^*(M,\,\mathsf{d},m)$ put

$$\nu_{i,h} := Q'_h(\nu_i) = \rho_{i,h} m_h$$

with $\rho_{i,h}(y) = \int \rho_i(x)Q'_h(y,dx)$ for $i = 0,1$ and let η_h be an h-midpoint of $\nu_{0,h}$ and $\nu_{1,h}$ such that

$$\mathrm{Ent}(\eta_h|m_h) \le \frac{1}{2}\mathrm{Ent}(\nu_{0,h}|m_h) + \frac{1}{2}\mathrm{Ent}(\nu_{1,h}|m_h) - \frac{K_h}{8}\,\mathsf{d}_W^{\delta_h h}(\nu_{0,h},\nu_{1,h})^2, \tag{1.2.8}$$

where δ_h is the sign of K_h.

From (1.2.5) – (1.2.7) we conclude

$$\begin{aligned} \mathsf{d}_W^2(\nu_0,\nu_{0,h}) &\le \frac{2 + \hat{\Delta}^2 \cdot \mathrm{Ent}(\nu_0|m)}{-\log \mathbb{D}((M,\,\mathsf{d},m),(M_h,\,\mathsf{d}_h,m_h))} \\ &\le \frac{2 + 4\Delta^2 R}{-\log \mathbb{D}((M,\,\mathsf{d},m),(M_h,\,\mathsf{d}_h,m_h))} \le \epsilon^2 \end{aligned}$$

and similarly $\mathsf{d}_W^2(\nu_1,\nu_{1,h}) \le \epsilon^2$.

If $K < 0$ we can suppose $K_h < 0$ too. From Lemma 1.2.5 (ii) we have

$$\mathsf{d}_W^{-h}(\nu_{0,h}, \nu_{1,h})^2 \le (\mathsf{d}_W(\nu_{0,h}, \nu_{1,h}) + h)^2 \le (\mathsf{d}_W(\nu_0, \nu_1) + 3\epsilon)^2 \le \mathsf{d}_W(\nu_0, \nu_1)^2 + 6\epsilon\Delta + 9\epsilon^2,$$

because $\mathsf{d}_W(\nu_0, \nu_1) \le \Delta$.

For $K > 0$ one can choose h small enough to ensure $K_h > 0$. Then Lemma 1.2.5 (i) implies

$$\mathsf{d}_W(\nu_0, \nu_1)^2 \le (\mathsf{d}_W(\nu_{0,h}, \nu_{1,h}) + 2\epsilon)^2 \le \left(\mathsf{d}_W^{+h}(\nu_{0,h}, \nu_{1,h}) + 3\epsilon\right)^2 \le \mathsf{d}_W^{+h}(\nu_0, \nu_1)^2 + 6\epsilon\Delta + 9\epsilon^2.$$

In both cases the estimates above combined with (1.2.6), (1.2.8) and the fact that we chose h with $-K_h < \epsilon - K$ will imply

$$\mathrm{Ent}(\eta_h | m_h) \le \frac{1}{2}\mathrm{Ent}(\nu_0 | m) + \frac{1}{2}\mathrm{Ent}(\nu_1 | m) - \frac{K}{8}\,\mathsf{d}_W^2(\nu_0, \nu_1) + \epsilon' \tag{1.2.9}$$

with $\epsilon' = \epsilon[\Delta^2 + 3|K|(2\Delta + 3\epsilon)]/8$.

The case $K = 0$ follows by the calculations above, depending on the sign of K_h.

Finally, put

$$\eta = Q_h(\eta_h).$$

Then again by (1.2.5), the estimates given in Lemma 4.19 [St06a] for Q_h and by the previous estimate (1.2.9) for $\mathrm{Ent}(\eta_h | m_h)$ we deduce

$$\begin{aligned} \mathsf{d}_W^2(\eta_h, \eta) &\le \frac{2 + \hat{\Delta}^2 \cdot \mathrm{Ent}(\eta_h | m_h)}{-\log \mathbb{D}((M, \mathsf{d}, m), (M_h, \mathsf{d}_h, m_h))} \\ &\le \frac{2 + 4\Delta^2 R}{-\log \mathbb{D}((M, \mathsf{d}, m), (M_h, \mathsf{d}_h, m_h))} \le \epsilon^2. \end{aligned}$$

For $i = 0, 1$ we have $\mathsf{d}_W(\eta, \nu_i) \le 2\epsilon + \mathsf{d}_W(\eta_h, \nu_{i,h}) \le 2\epsilon + \frac{1}{2}\mathsf{d}_W(\nu_{0,h}, \nu_{1,h}) + h \le \frac{1}{2}\mathsf{d}_W(\nu_0, \nu_1) + 4\epsilon$. Hence,

$$\sup_{i=0,1} \mathsf{d}_W(\eta, \nu_i) \le \frac{1}{2}\mathsf{d}_W(\nu_0, \nu_1) + 4\epsilon,$$

i.e. η is a (4ϵ)-midpoint of ν_0 and ν_1. Furthermore, by (1.2.6)

$$\begin{aligned} \mathrm{Ent}(\eta | m) &\le \mathrm{Ent}(\eta_h | m_h) \\ &\le \frac{1}{2}\mathrm{Ent}(\nu_0 | m) + \frac{1}{2}\mathrm{Ent}(\nu_1 | m) - \frac{K}{8}\,\mathsf{d}_W^2(\nu_0, \nu_1) + \epsilon' \end{aligned}$$

with ϵ' as above. This proves that $\mathbb{C}\mathrm{urv}_{lax}(M, \mathsf{d}, m) \ge K$. □

1.3 Discretizations of metric measure spaces

Let (M, d, m) be a given metric measure space. For $h > 0$ let M_h be a discrete subset of M, say $M_h = \{x_n : n \in \mathbb{N}\}$, with $M = \bigcup_{i=1}^{\infty} B_R(x_i)$, where $R = R(h) \searrow 0$ as $h \searrow 0$. If (M, d, m) has finite diameter then M_h might consist of a finite number of points. Choose $A_i \subset B_R(x_i)$ mutually disjoint with $x_i \in A_i$, $i = 1, 2, \dots$ and $\bigcup_{i=1}^{\infty} A_i = M$ (e.g. one could choose a Voronoi tessellation) and consider the measure m_h on M_h given by $m_h(\{x_i\}) := m(A_i)$, $i = 1, 2, \dots$. We call (M_h, d, m_h) a discretization of (M, d, m).

Theorem 1.3.1. *(i) If $m(M) < \infty$ then $(M_h, \mathsf{d}, m_h) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$ as $h \to 0$.*

(ii) If $\mathbb{C}\mathrm{urv}_{lax}(M, \mathsf{d}, m) \geq K$ with $K \neq 0$ then for each $h > 0$ and for each discretization (M_h, d, m_h) with $R(h) < h/4$ we have h-$\mathbb{C}\mathrm{urv}(M_h, \mathsf{d}, m_h) \geq K$.

(iii) If $\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \geq K$ for some real number K then for each $h > 0$ and for each discretization (M_h, d, m_h) with $R(h) \leq h/4$ we have h-$\mathbb{C}\mathrm{urv}(M_h, \mathsf{d}, m_h) \geq K$.

Proof. (i) The measure $q = \sum_{i=1}^{\infty} (m(A_i)\delta_{x_i}) \times (1_{A_i} m)$ is a coupling of m_h and m, so

$$\begin{aligned}
\mathbb{D}^2((M_h, \mathsf{d}, m_h), (M, \mathsf{d}, m)) &\leq \int_{M_h \times M} \mathsf{d}^2(x, y)\, dq(x, y) \\
&= \sum_{i=1}^{\infty} m(A_i) \int_{A_i} \mathsf{d}^2(x_i, y)\, dm(y) \\
&\leq \left(\sum_{i=1}^{\infty} m(A_i)^2 \right) R(h)^2 \leq R(h)^2 \left(\sum_{i=1}^{\infty} m(A_i) \right)^2 \\
&= R(h)^2 m(M)^2 \to 0 \text{ as } h \to 0.
\end{aligned}$$

(ii) Fix $h > 0$ and consider a discretization (M_h, d, m_h) of (M, d, m) with $R(h) < h/4$. Let $\nu_0^h, \nu_1^h \in \mathcal{P}_2^*(M_h, \mathsf{d}, m_h)$ be given; it is enough to make the proof for ν_0^h, ν_1^h with compact support. Suppose then $\nu_i^h = \left(\sum_{j=1}^{n} \alpha_{i,j}^h 1_{\{x_j\}} \right) m_h$, $i = 1, 2$ (some of the $\alpha_{i,j}^h$ can be zero). We take also an arbitrary $t \in [0, 1]$. Put $\nu_i := \left(\sum_{j=1}^{n} \alpha_{i,j}^h 1_{A_j} \right) m \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ for $i = 1, 2$. Choose $\epsilon > 0$ such that

$$4R(h) + \epsilon \leq h. \tag{1.3.1}$$

Since $\mathbb{C}\mathrm{urv}_{lax}(M, \mathsf{d}, m) \geq K$ for our given $t \in [0, 1]$ there exists $\eta_t \in \mathcal{P}_2^*(M, \mathsf{d}, m)$ an ϵ-rough t-intermediate point between ν_0 and ν_1 such that

$$\mathrm{Ent}(\eta_t | m) \leq (1-t)\mathrm{Ent}(\nu_0 | m) + t\mathrm{Ent}(\nu_1 | m) - \frac{K}{2} t(1-t)\, \mathsf{d}_W^2(\nu_0, \nu_1) + \epsilon. \tag{1.3.2}$$

We compute

$$\mathrm{Ent}(\nu_i|m) = \sum_{j=1}^{n} \int_{A_j} \alpha_{i,j}^h \log \alpha_{i,j}^h \, dm = \sum_{j=1}^{n} \alpha_{i,j}^h \log \alpha_{i,j}^h \, m_h(\{x_j\}) = \mathrm{Ent}(\nu_i^h|m_h), \tag{1.3.3}$$

for $i = 0, 1$. Denote $\eta_t^h(\{x_j\}) := \eta_t(A_j)$, $j = 1, 2, ..., n$. Suppose $\eta_t = \rho_t \cdot m$. From Jensen's inequality we get

$$\begin{aligned} \mathrm{Ent}(\eta_t^h|m_h) &= \sum_{j=1}^{\infty} \frac{\int_{A_j} \rho_t \, dm}{m(A_j)} \log \frac{\int_{A_j} \rho_t \, dm}{m(A_j)} m_h(\{x_j\}) \\ &\leq \sum_{j=1}^{\infty} \left(\frac{1}{m(A_j)} \int_{A_j} \rho_t \log \rho_t \, dm \right) m_h(\{x_j\}) = \mathrm{Ent}(\eta_t|m), \end{aligned}$$

which together with (1.3.2) and (1.3.3) implies

$$\mathrm{Ent}(\eta_t^h|m_h) \leq (1-t)\mathrm{Ent}(\nu_0^h|m_h) + t\mathrm{Ent}(\nu_1^h|m_h) - \frac{K}{2} t(1-t) \, \mathsf{d}_W^2(\nu_0, \nu_1) + \epsilon. \tag{1.3.4}$$

Firstly, we consider the case $K < 0$. Let q^h be a $-2R(h)$-optimal coupling of ν_0^h and ν_1^h. Then the formula

$$\widehat{q} := \sum_{j,k=1}^{n} \left[q^h(\{(x_j, x_k)\}) \delta_{(x_j, x_k)} \times \frac{1_{A_j \times A_k}}{m(A_j) m(A_k)} (m \times m) \right]$$

defines a measure on $M_h \times M_h \times M \times M$ which has marginals ν_0^h, ν_1^h, ν_0 and ν_1. Moreover, the projection of $\widehat{q}$ on the first two factors is equal to q^h. Therefore we have

$$\begin{aligned} \mathsf{d}_W(\nu_0, \nu_1)^2 &\leq \int \mathsf{d}(x, y)^2 d\widehat{q}(x^h, y^h, x, y) \\ &\leq \int \left[\mathsf{d}(x, x^h) + \mathsf{d}(x^h, y^h) + \mathsf{d}(y^h, y) \right]^2 d\widehat{q}(x^h, y^h, x, y) \\ &= \sum_{j,k=1}^{n} \frac{q^h(\{(x_j, x_k)\})}{m(A_j) m(A_k)} \int_{A_j \times A_k} \left[\mathsf{d}(x, x_j) + \mathsf{d}(x_j, x_k) \right. \\ &\qquad\qquad \left. + \mathsf{d}(x_k, y) \right]^2 dm(x) dm(y) \\ &\leq \sum_{j,k=1}^{n} q^h(\{(x_j, x_k)\}) \left(\mathsf{d}(x_j, x_k) + 2R(h) \right)^2 = \mathsf{d}_W^{-2R(h)}(\nu_0^h, \nu_1^h)^2, \end{aligned}$$

which together with (1.3.4) yields

$$\mathrm{Ent}(\eta_t^h|m_h) \leq (1-t)\mathrm{Ent}(\nu_0^h|m_h) + t\mathrm{Ent}(\nu_1^h|m_h) - \frac{K}{2} t(1-t) \, \mathsf{d}_W^{-2R(h)}(\nu_0^h, \nu_1^h)^2 + \epsilon. \tag{1.3.5}$$

In the case $K > 0$ we start with an optimal coupling q of ν_0 and ν_1 and we show that the measure

$$\widetilde{q}^h := \sum_{j,k=1}^{n} q(A_j \times A_k)\delta_{(x_j,x_k)}$$

is a coupling of ν_0^h and ν_1^h. Indeed, if $A \subset M_h$ then we have in turn

$$\begin{aligned}
\sum_{j,k=1}^{n} q(A_j \times A_k)\delta_{(x_j,x_k)}(A \times M_h) &= \sum_{j,k=1}^{n} q(A_j \times A_k)\delta_{x_j}(A) = \sum_{j=1}^{n} q(A_j \times M)\delta_{x_j}(A) \\
&= \sum_{j=1}^{n} \nu_0(A_j)\delta_{x_j}(A) = \sum_{j=1}^{n} \nu_0^h(\{x_j\})\delta_{x_j}(A) \\
&= \nu_0^h(A).
\end{aligned}$$

Since for any $j,k = 1,2,\ldots.n$ and for arbitrary $x \in A_j$ and $y \in A_k$ we have $(\mathsf{d}(x_j,x_k) - 2R(h))_+ \leq (\mathsf{d}(x_j,x_k) - \mathsf{d}(x,x_j) - \mathsf{d}(y,x_k))_+ \leq \mathsf{d}(x,y)$ one can estimate:

$$\begin{aligned}
\mathsf{d}_W^{+2R(h)}(\nu_0^h,\nu_1^h)^2 &\leq \sum_{j,k=1}^{n} q(A_j \times A_k)\left[(\mathsf{d}(x_j,x_k) - 2R(h))_+\right]^2 \\
&= \sum_{j,k=1}^{n} \int_{A_j \times A_k} \left[(\mathsf{d}(x_j,x_k) - 2R(h))_+\right]^2 dq(x,y) \\
&\leq \sum_{j,k=1}^{n} \int_{A_j \times A_k} \left[(\mathsf{d}(x_j,x_k) - \mathsf{d}(x,x_j) - \mathsf{d}(y,x_k))_+\right]^2 dq(x,y) \\
&\leq \sum_{j,k=1}^{n} \int_{A_j \times A_k} \mathsf{d}(x,y)^2 dq(x,y) = \int_{M \times M} \mathsf{d}(x,y)^2 dq(x,y) \\
&= \mathsf{d}_W(\nu_0,\nu_1)^2.
\end{aligned}$$

Therefore from (1.3.4) we obtain

$$\mathrm{Ent}(\eta_t^h|m_h) \leq (1-t)\mathrm{Ent}(\nu_0^h|m_h) + t\mathrm{Ent}(\nu_1^h|m_h) - \frac{K}{2}t(1-t)\,\mathsf{d}_W^{+2R(h)}(\nu_0^h,\nu_1^h)^2 + \epsilon. \tag{1.3.6}$$

For ϵ sufficiently small we can get

$$-\frac{K}{2}t(1-t)\,\mathsf{d}_W^{\pm 2R(h)}(\nu_0^h,\nu_1^h)^2 + \epsilon \leq -\frac{K}{2}t(1-t)\,\mathsf{d}_W^{\pm h}(\nu_0^h,\nu_1^h)^2 \tag{1.3.7}$$

and then (1.3.5), (1.3.6) yield

$$\mathrm{Ent}(\eta_t^h|m_h) \leq (1-t)\mathrm{Ent}(\nu_0^h|m_h) + t\mathrm{Ent}(\nu_1^h|m_h) - \frac{K}{2}t(1-t)\,\mathsf{d}_W^{\pm h}(\nu_0^h,\nu_1^h)^2, \tag{1.3.8}$$

depending on the sign of K. The inequality (1.3.7) fails only when $K > 0$ and $\mathsf{d}_W^{+h}(\nu_0^h, \nu_1^h) = 0$, but in this case $\mathsf{d}_W(\nu_0^h, \nu_1^h) \leq h$ and either $\eta = \nu_0^h$ or $\eta = \nu_1^h$ verifies directly the condition (1.2.3) from the definition of h-rough curvature bound for the discretization.

The measure $\pi = \sum_{j=1}^n \left(\eta_t^h(\{x_j\})\delta_{x_j} \times 1_{A_j}\eta_t\right)$ is a coupling of η_t^h and η_t, so

$$\mathsf{d}_W^2(\eta_t^h, \eta_t) \leq \int_{M_h \times M} \mathsf{d}^2(x, y)\, d\pi(x, y) \leq R^2(h),$$

and similarly $\mathsf{d}_W^2(\nu_i^h, \nu_i) \leq R^2(h)$ for $i = 1, 2$. Because η_t is an ϵ-rough t-intermediate point between ν_0 and ν_1 we deduce

$$\begin{aligned} \mathsf{d}_W(\eta_t^h, \nu_0^h) &\leq \mathsf{d}_W(\eta_t, \nu_0) + 2R(h) \leq t\, \mathsf{d}_W(\nu_0, \nu_1) + 2R(h) + \epsilon \\ &\leq t\, \mathsf{d}_W(\nu_0^h, \nu_1^h) + 2R(h)(1 + t) + \epsilon \end{aligned}$$

and by a similar argument

$$\mathsf{d}_W(\eta_t^h, \nu_1^h) \leq (1 - t)\, \mathsf{d}_W(\nu_0^h, \nu_1^h) + 2R(h)(2 - t) + \epsilon.$$

From (1.3.1) we conclude that η^h is an h-rough t-intermediate point between ν_0^h and ν_1^h, which together with (1.3.8) proves that $h\text{-}\mathbb{C}\mathrm{urv}(M_h, \mathsf{d}, m_h) \geq K$.

(iii) follows the same lines as (ii). □

Example 1.3.2. (i) If we consider on $\mathbb{Z}^n$ the metric d_1 coming from the norm $|\cdot|_1$ in $\mathbb{R}^n$ defined by $|x|_1 = \sum_{i=1}^n |x_i|$ and with the measure $\overline{m}_n = \sum_{x \in \mathbb{Z}^n} \delta_x$, then $h\text{-}\mathbb{C}\mathrm{urv}(\mathbb{Z}^n, \mathsf{d}_1, \overline{m}_n) \geq 0$ for any $h \geq 2n$.

(ii) The n-dimensional grid $\mathbb{E}^n$ having $\mathbb{Z}^n$ as set of vertices, equipped with the graph distance and with the measure m_n which is the 1-dimensional Lebesgue measure on the edges, has $h\text{-}\mathbb{C}\mathrm{urv}(\mathbb{E}^n, \mathsf{d}_1, m_n) \geq 0$ for any $h \geq 2(n + 1)$.

Proof. We use the following result:

Lemma 1.3.3. [CE05] *Any finite dimensional Banach space that is equipped with the Lebesgue measure has curvature* ≥ 0.

We tile the space $\mathbb{R}^n$ with n-dimensional cubes of edge 1 centered in the vertices of the grid. The $|\cdot|_1$-radius of the cells of the tessellation with such cubes is $n/2$. Therefore, claim (i) is a consequence of Theorem 1.3.1(iii) applied to the space $(\mathbb{R}^n, |\cdot|_1, dx)$ and of Lemma 1.3.3.

For the proof of (ii) we follow the same argument like in the proof of Theorem 1.3.1. In this case, we pass from a probability on the grid to a probability on $\mathbb{R}^n$

by averaging on each cube of the tessellation and scaling. Here one should take into account that for a cube C from the tiling

$$\sup\{|x-y|_1 : x \in C \cap \mathbb{E}^n, y \in C\} = \frac{n+1}{2},$$

that provides the minimal $h = 2(n+1)$ starting from which h- $\mathbb{C}\mathrm{urv}(\mathbb{E}^n, \mathsf{d}_1, m_n) \geq 0$. $\square$

Example 1.3.4. (i) Let G be the graph that tiles the euclidian plane with equilateral triangles of edge r. We endow G with the graph metric d_G induced by the euclidian metric and with the 1-dimensional Lebesgue measure m on the edges. Then G has h-curvature ≥ 0 for any $h \geq 8r\sqrt{3}/3$.

(ii) The graph G' that tiles the euclidian plane with regular hexagons of edge length r, equipped as usual with the graph metric $\mathsf{d}_{G'}$ and with the 1-dimensional measure m', has h-curvature ≥ 0 for any $h \geq 34r/3$.

Proof. Consider a cartesian coordinate system in the euclidian plane with origin O and axes Ox and Oy. We equip $\mathbb{R}^2$ with the Banach norm $\|\cdot\|$ that has as unit ball the regular hexagon centered in O, having two opposite vertices on Ox and the edge length (measured in the euclidian metric) equal to 1. Explicitly $\|(x,y)\| = \max\{\frac{2\sqrt{3}}{3}|y|, |x| + \frac{\sqrt{3}}{3}|y|\}$ for any (x,y) in $\mathbb{R}^2$. We denote by d the metric determined by this norm.

(i) For the triangular tessellation we choose the origin O to be one of the vertices of the graph and two of the 6 edges emanating from O be along the Ox axis. The edges of the graph have length r in the euclidian metric. We see that $\mathsf{d}_G(v_1, v_2) = \mathsf{d}(v_1, v_2)$ for any two vertices v_1 and v_2 of the graph. In general for $x, y \in G$ we have $|\,\mathsf{d}_G(x,y) - \mathsf{d}(x,y)| \leq r$. Then one can construct a coupling $\widehat{\mathsf{d}}$ of d_G and d by setting $\widehat{\mathsf{d}}(v,x) := \mathsf{d}(v,x)$ for v vertex of G and $x \in \mathbb{R}^2$ and $\widehat{\mathsf{d}}(y,x) := \inf_{i=1,2}\{\,\mathsf{d}_G(y,v_i) + \mathsf{d}(v_i,x))\}$ if $y \in G$ belongs to an edge with endpoints v_1, v_2 and $x \in \mathbb{R}^2$.

By Lemma 1.3.3 $\mathbb{C}\mathrm{urv}(\mathbb{R}^2, \mathsf{d}, \lambda) \geq 0$, where λ is de 2-dimensional Lebesgue measure. If we tile the plane with regular hexagons A_j, $j \in \mathbb{N}$, which have vertices in the centers of the triangles of the graph G, we have $\widehat{\mathsf{d}}(y,x) \leq 2r\sqrt{3}/3$ for any $y \in A_j \cap G$ and $x \in A_j$. The proof of the h-curvature bound is a modification of the proof of Theorem 1.3.1. We start with $\nu_0, \nu_1 \in \mathcal{P}_2^*(G, \mathsf{d}_G, m)$ with $\nu_i = \rho_i m$, $i = 0, 1$ and we define

$$\widetilde{\nu}_i := \sum_{j=1}^{\infty} \frac{1}{\lambda(A_j)} \left(\int_{G \cap A_j} \rho_i \, dm \right) 1_{A_j} \cdot \lambda \in \mathcal{P}_2^*(\mathbb{R}^2, \mathsf{d}, \lambda) \text{ for } i = 0, 1.$$

We have then $\widehat{\mathsf{d}}_W(\nu_i, \widetilde{\nu}_i) \leq 2r\sqrt{3}/3$. We consider $\widetilde{\eta}_t = \widetilde{\rho}_t \cdot \lambda$ the geodesic that joints $\widetilde{\nu}_0$ and $\widetilde{\nu}_1$, along which the convexity condition for the entropy on $\mathcal{P}_2^*(\mathbb{R}^2, \mathsf{d}, \lambda)$ is fulfilled and denote

$$\eta_t := \sum_{j=1}^{\infty} \frac{1}{m(G \cap A_j)} \left(\int_{A_j} \widetilde{\rho}_t \, d\lambda \right) 1_{G\cap A_j} \cdot m.$$

Then η_t is $8r\sqrt{3}/3$-rough t-intermediate point between ν_0 and ν_1. From Jensen's inequality we obtain $\mathrm{Ent}(\eta_t|m) \leq \mathrm{Ent}(\widetilde{\eta}_t|\lambda) - \log m(G\cap A) + \log \lambda(A)$ and $\mathrm{Ent}(\widetilde{\nu}_i|\lambda) \leq \mathrm{Ent}(\nu_i|m) + \log m(G\cap A) - \log \lambda(A)$ (observe that all sets A_j have the same Lebesgue measure $\lambda(A)$ and all sets $G\cap A_j$ have the same measure $m(G\cap A)$). Hence η_t satisfies

$$\mathrm{Ent}(\eta_t|m) \leq (1-t)\mathrm{Ent}(\nu_0|m) + t\mathrm{Ent}(\nu_1|m),$$

and so we have proved $h\text{-}\mathbb{C}\mathrm{urv}(G, \mathsf{d}_G, m) \geq 0$ for any $h \geq 8r\sqrt{3}/3$.

(ii) For the hexagonal tessellation let O be again one of the vertices of the graph and one of the 3 edges emanating from it be along the Oy axis. In this case we use the Banach norm $\|\cdot\|' := \frac{3}{4}\|\cdot\|$ on $\mathbb{R}^2$ and denote by d' the associated metric. The length of the edges of the graph in the metric d' is equal to $4r/3$. We see that $\mathsf{d}_{G'}(v_1, v_2) = \mathsf{d}'(v_1, v_2)$ for any two vertices v_1, v_2 with $\mathsf{d}_{G'}(v_1, v_2) = 2kr$, $k \in \mathbb{N}$. In general $|\,\mathsf{d}_{G'} - \mathsf{d}'| \leq r/3$ on the set of vertices and $|\,\mathsf{d}_{G'} - \mathsf{d}'| \leq r$ everywhere on G'.

One can construct then a coupling $\widehat{\mathsf{d}'}$ of $\mathsf{d}_{G'}$ and d' in the following way: Fix $v_0 = O$. If v is a vertex of the graph with $\mathsf{d}_{G'}(v_0, v) = 2kr$, $k \in \mathbb{N}$ then set $\widehat{\mathsf{d}'}(v, x) := \mathsf{d}'(v, x)$, $x \in \mathbb{R}^2$. For $y \in G'$ with $\mathsf{d}_{G'}(v_0, y) \neq 2kr$, $k \in \mathbb{N}$ define $\widehat{\mathsf{d}'}(y, x) := \inf\{\,\mathsf{d}_{G'}(y, v) + \mathsf{d}'(v, x) : v \in G',\ \mathsf{d}_{G'}(v_0, v) = 2kr\}$.

We tile the plane with equilateral triangles B_i, $i \in \mathbb{N}$, with vertices in the centers of the hexagons of the graph. Then $\widehat{\mathsf{d}'}(y, x) \leq 17r/6$ for $y \in B_i \cap G'$, $x \in B_i$. By the same argument as for the triangular tiling we obtain $h\text{-}\mathbb{C}\mathrm{urv}(G', \mathsf{d}_{G'}, m') \geq 0$ for any $h \geq 4 \cdot 17r/6 = 34r/3$. $\square$

1.4 Some remarks on homogeneous planar graphs

We refer in the sequel to a special class of graphs. In general, a graph G is determined by the set of vertices $V(G)$ and the set of edges $E(G)$. In order to regard graphs as discrete analogues of 2-dimensional manifolds one has to specify also the set of faces $F(G)$ and to impose the graph to be planar. A graph is planar if it can be drawn in a plane without graph edges crossing (i.e., it has graph crossing number 0). Only planar graphs have duals. The graphs we will be concerned with are connected and simple (with no self-loops and no multiple edges) and such that their dual graphs are also simple, therefore any two faces have at most one common edge and every face is bounded by a cycle.

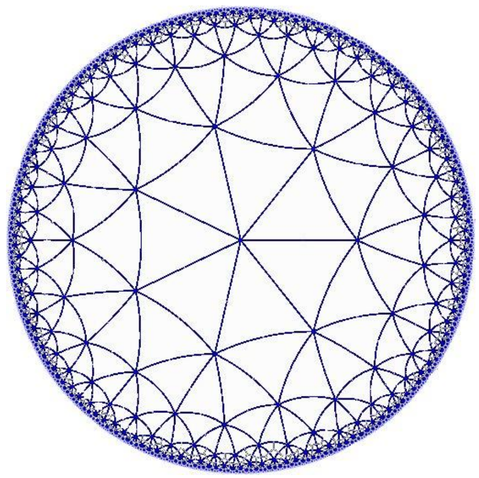

Figure 1.1: $\mathbb{G}(7,3,r)$

We consider in the following the (possibly infinite) homogeneous graph $\mathbb{G}(l,n,r)$ with vertices of constant degree $l \geq 3$, with faces bounded by polygons with $n \geq 3$ edges (thus n is the degree of all vertices in the dual graph) and such that all edges have the same length $r > 0$.

The following result is probably well-known, but since we didn't find a reference we present here the easy proof.

Lemma 1.4.1. *(i) If $\frac{1}{l} + \frac{1}{n} < \frac{1}{2}$ then $\mathbb{G}(l,n,r)$ can be embedded into the 2-dimensional hyperbolic space with constant sectional curvature*

$$K = -\frac{1}{r^2}\left[\operatorname{arccosh}\left(2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1\right)\right]^2. \tag{1.4.1}$$

There are infinitely many choices of such l and n. In any case, the graph is unbounded.

(ii) If $\frac{1}{l} + \frac{1}{n} > \frac{1}{2}$ then $\mathbb{G}(l,n,r)$ is one of the five regular polyhedra (Tetrahedron, Octahedron, Cube, Icosahedron, Dodecahedron) and can be embedded into the 2-dimensional sphere with constant sectional curvature

$$K = \frac{1}{r^2}\left[\arccos\left(2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1\right)\right]^2. \tag{1.4.2}$$

(iii) If $\frac{1}{l} + \frac{1}{n} = \frac{1}{2}$ then $\mathbb{G}(l,n,r)$ can be embedded into the euclidian plane ($K = 0$). In this case there are exactly three cases corresponding to the 3 regular tessellations of the euclidian plane: the tessellation of triangles ($l = 6$, $n = 3$), of squares ($l = n = 4$), and of hexagons ($l = 3$, $n = 6$).

Proof. Firstly we see that

$$2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1 > 1 \Leftrightarrow \sin^2\left(\frac{\pi}{2} - \frac{\pi}{n}\right) > \sin^2\left(\frac{\pi}{l}\right) \Leftrightarrow \frac{1}{l} + \frac{1}{n} < \frac{1}{2}$$

hence in each case the expression that defines the curvature K makes sense.

(i) For given l, n, r we construct the embedding in the following way: we start from an arbitrary point O of the 2-hyperbolic space with curvature K, denoted by $\mathbb{H}^{K,2}$. From this point we construct n geodesic lines $OA_1, OA_2, \cdots, OA_n$ of length

$$R := \frac{1}{\sqrt{-K}}\operatorname{arcsinh}\left(\frac{\sinh\left(\sqrt{-K}r\right)}{\sin\left(\frac{2\pi}{n}\right)}\sin\left(\frac{\pi}{l}\right)\right), \tag{1.4.3}$$

such that the inner angle between any two consecutive geodesics OA_k, OA_{k+1} is $2\pi/n$. We prove that $A_1, A_2, \cdots, A_n$ correspond to vertices of the given graph, and the geodesics $A_1A_2, \cdots, A_{n-1}A_n, A_nA_1$ correspond isometrically to consecutive edges in $\mathbb{G}(l, n, r)$ that bound a regular n-polygon with edge-length r and all angles equal to $2\pi/l$. Let us denote by d the intrinsic metric on $\mathbb{H}^{K,2}$.

From the Cosine Rule for hyperbolic triangles applied to ΔOA_1A_2 and from (1.4.1) and (1.4.3) we have:

$$\begin{aligned}
\cosh\left(\sqrt{-K}\,\mathsf{d}(A_1, A_2)\right) &= \cosh^2(\sqrt{-K}R) - \sinh^2(\sqrt{-K}R)\cos\left(\frac{2\pi}{n}\right) \\
&= 1 + \sinh^2(\sqrt{-K}R)\left(1 - \cos\left(\frac{2\pi}{n}\right)\right) \\
&= 1 + \frac{\sinh^2\left(\sqrt{-K}r\right)}{\sin^2\left(\frac{2\pi}{n}\right)}\sin^2\left(\frac{\pi}{l}\right)\left(1 - \cos\left(\frac{2\pi}{n}\right)\right) \\
&= 1 + \frac{\cosh^2\left(\sqrt{-K}r\right) - 1}{1 + \cos\left(\frac{2\pi}{n}\right)}\sin^2\left(\frac{\pi}{l}\right) \\
&= 1 + \frac{\sin^2\left(\frac{\pi}{l}\right)}{2\cos^2\left(\frac{\pi}{n}\right)}\left[\left(2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1\right)^2 - 1\right] \\
&= 2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1 = \cosh(\sqrt{-K}r),
\end{aligned}$$

so $\mathsf{d}(A_1, A_2) = r$ and the same holds for all the other edges of the polygon. We apply now the Sine Rule for the hyperbolic triangle ΔOA_1A_2 and (1.4.3) in order to compute:

$$\sin\sphericalangle(A_1; O, A_2) = \frac{\sin\left(\frac{2\pi}{n}\right)}{\sinh(\sqrt{-K}r)}\sinh(\sqrt{-K}R) = \sin\left(\frac{\pi}{l}\right), \tag{1.4.4}$$

where $\sphericalangle(A_1; O, A_2)$ denotes the angle at A_1 in the triangle $\Delta O A_1 A_2$. This angle is less then $\pi/2$ because it is equal to $\sphericalangle(A_2; O, A_1)$ and in the hyperbolic triangles the sum of the angles of a triangle is less then π. Therefore (1.4.4) shows that all the angles of the polygon are equal to $2\pi/l$, so around each vertex one can construct other $l-1$ polygons with n edges, congruent with the first one. We repeat the procedure with each of the vertices of the new polygons. In this way the whole space $\mathbb{H}^{K,2}$ can be tiled with regular polygons which are faces of the graph $\mathbb{G}(l,n,r)$.

(ii), (iii) Since there is only a finite number of examples with well-known realizations, the claim can be verified directly. Alternatively, one can prove it like in the part (i) with appropriate interpretations of the hyperbolic sine as sine for positive curvature and as length for the euclidian plane.

□

Remark 1.4.2. The dual graph $\mathbb{G}(l,n,r)^* = \mathbb{G}(n,l,r')$ is embedded into the 2-manifold of the same constant curvature as $\mathbb{G}(l,n,r)$, where the dual edge length is

$$r' := r \cdot \operatorname{arccosh}\left(2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1\right) \Big/ \operatorname{arccosh}\left(2\frac{\cos^2\left(\frac{\pi}{l}\right)}{\sin^2\left(\frac{\pi}{n}\right)} - 1\right) \quad \text{for } K < 0$$

and with appropriate modifications for the other two cases.

In each of the three cases from Lemma 1.4.1 the 2-manifold will be endowed with the intrinsic metric d and with the Riemannian volume vol. We equip $\mathbb{G}(l,n,r)$ with the metric d induced by the corresponding Riemannian metric and with the uniform measure m on the edges. We denote further by $\mathbb{V}(l,n,r)$ the set of vertices of the graph $\mathbb{G}(l,n,r)$ equipped with the same metric d inherited from the Riemannian manifold and with the counting measure $\widetilde{m} := \sum_{v\in\mathbb{V}} \delta_v$.

Theorem 1.4.3. *For any numbers $l, n \geq 3$ and for any $r > 0$ both metric measure spaces $(\mathbb{V}(l,n,r), \mathsf{d}, \widetilde{m})$ and $(\mathbb{G}(l,n,r), \mathsf{d}, m)$ have h-curvature $\geq K$ for $h \geq r \cdot C(l,n)$, where*

$$K = \begin{cases} -\frac{1}{r^2}\left[\operatorname{arccosh}\left(2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1\right)\right]^2 & \text{for } \frac{1}{l} + \frac{1}{n} > \frac{1}{2} \\ \frac{1}{r^2}\left[\arccos\left(2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1\right)\right]^2 & \text{for } \frac{1}{l} + \frac{1}{n} < \frac{1}{2} \\ 0 & \text{for } \frac{1}{l} + \frac{1}{n} = \frac{1}{2} \end{cases} \tag{1.4.5}$$

and $C(l,n) = 4 \cdot \operatorname{arcsinh}\left(\frac{1}{sin\left(\frac{\pi}{n}\right)}\sqrt{\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1}\right) \Big/ \operatorname{arccosh}\left(2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1\right).$

Proof. We look at $\mathbb{V}(l,n,r)$ and $\mathbb{G}(l,n,r)$ as subsets of the 2-manifold with constant curvature K (given by Lemma 1.4.1). We tile the manifold with the faces of the dual graph $\mathbb{G}(n,l,r')$ having vertices in the centers of the faces of $\mathbb{G}(l,n,r)$ (the center O of the polygon with n edges in the proof of Lemma 1.4.1 becomes vertex of the dual).

We make explicitly the calculations only in the hyperbolic case, the other two cases are similar. One can decompose the hyperbolic space as $\mathbb{H}^{K,2} = \bigcup_{j=1}^{\infty} F_j$, where $\{F_j\}_j$ are the faces of the dual graph, as described above. The curvature bound for the discrete space $\mathbb{V}(l,n,r)$ is then a consequence of the Theorem 1.3.1. For $\mathbb{G} := \mathbb{G}(l,n,r)$ the proof of the curvature bound is a modification of the proof of Theorem 1.3.1. We start with $\nu_0, \nu_1 \in \mathcal{P}_2^*(\mathbb{G}(l,n,r), \mathsf{d}, m)$ with $\nu_i = \rho_i \cdot m$, $i = 0,1$ and define

$$\widetilde{\nu_i} := \sum_{j=1}^{\infty} \frac{1}{\mathrm{vol}(F_j)} \left(\int_{\mathbb{G}\cap F_j} \rho_i \, dm \right) 1_{F_j} \cdot \mathrm{vol} \in \mathcal{P}_2^*(\mathbb{H}^{K,2}, \mathsf{d}, \mathrm{vol}) \text{ for } i = 0,1.$$

Now the place of $R(h)$ from Theorem 1.3.1 is taken by R from the proof of Lemma 1.4.1(i), so $\mathsf{d}_W(\nu_i, \widetilde{\nu_i}) \leq R$. One can express R only in terms of our initial data l, n and r as $R = rC(l,n)/4$, with $C(l,n)$ given in the statement of the theorem. We consider $\widetilde{\eta_t} = \widetilde{\rho_t} \cdot \mathrm{vol}$ the geodesic that joints $\widetilde{\nu_0}$ and $\widetilde{\nu_1}$, along which one has the K-convexity for the entropy on $\mathbb{H}^{K,2}$ (Theorem 4.9 from [St06a]) and denote

$$\eta_t := \sum_{j=1}^{\infty} \frac{1}{m(\mathbb{G}\cap F_j)} \left(\int_{F_j} \widetilde{\rho_t} \, d\mathrm{vol} \right) 1_{\mathbb{G}\cap F_j} \cdot m.$$

Then η_t is $4R$-rough t-intermediate point between ν_0 and ν_1. From Jensen's inequality we obtain $\mathrm{Ent}(\eta_t|m) \leq \mathrm{Ent}(\widetilde{\eta_t}|\mathrm{vol}) - \log m(\mathbb{G}\cap F) + \log \mathrm{vol}(F)$ and $\mathrm{Ent}(\widetilde{\nu_i}|\mathrm{vol}) \leq \mathrm{Ent}(\nu_i|m) + \log m(\mathbb{G}\cap F) - \log \mathrm{vol}(F)$ (observe that all faces F_j have the same volume $\mathrm{vol}(F)$ and all sets $\mathbb{G}\cap F_j$ have the same measure $m(\mathbb{G}\cap F)$). Hence, like in the proof of Theorem 1.3.1, η_t satisfies

$$\mathrm{Ent}(\eta_t|m) \leq (1-t)\mathrm{Ent}(\nu_0|m) + t\mathrm{Ent}(\nu_1|m) - \frac{K}{2} t(1-t)\, \mathsf{d}_W^{-2R}(\nu_0,\nu_1)^2,$$

so we have proved $h\text{-}\mathrm{Curv}(\mathbb{G}(l,n,r), \mathsf{d}, m) \geq K$ for any $h \geq 4R$ in the hyperbolic case ($K < 0$).

□

Remark 1.4.4. There are various notions of combinatorial curvature for graphs in the literature, see for instance [Gro87], [Hi01], [Fo03]. The notion of curvature introduced by Gromov in [Gro87] was used in studying hyperbolic groups. Later on it was modified and investigated by Higuchi [Hi01] and other authors. Forman has introduced in [Fo03] a different notion of combinatorial Ricci curvature for cell complexes. The graphs considered in the above mentioned works have neither specified metric, nor specified reference measure.

In [Hi01] the combinatorial curvature of a graph G is a map $\Phi_G : V(G) \to \mathbb{R}$ that assigns to each vertex $x \in V(G)$ the number $\Phi_G(x) = 1 - \frac{m(x)}{2} + \sum_{i=1}^{m(x)} \frac{1}{d(F_i)}$, where $m(x)$ is the degree of the vertex x, $d(F)$ is the number of edges of the cycle bounding a face F, and $F_1, F_2, \ldots, F_{m(x)}$ are the faces around the vertex x. The combinatorial curvature introduced in [Gro87] is a map $\Phi^*_G : F(G) \to \mathbb{R}$, where the curvature $\Phi^*_G(F)$ of a face F is given by the curvature Φ_G of the corresponding vertex in the dual graph. For the homogeneous graph $\mathbb{G}(l, n, r)$, the curvature of any vertex x is $\Phi_G(x) = l(\frac{1}{l} + \frac{1}{n} - \frac{1}{2})$ and the curvature in the sense of Gromov [Gro87] of any face F is $\Phi^*_G(F) = n(\frac{1}{l} + \frac{1}{n} - \frac{1}{2})$.

Note that the sign of the combinatorial curvature in both approaches above changes according to whether $\frac{1}{l} + \frac{1}{n}$ is greater or less than $\frac{1}{2}$. In our Theorem 1.4.3 the sign of the rough curvature bound changes in the same manner, although our notion of curvature applies to graphs that have a metric structure and a reference measure. For the moment we see no further links with the notions of combinatorial curvature mentioned here.

1.5 Perturbed transportation inequalities, concentration of measure and exponential integrability

Let (M, d) be a metric space and $m \in \mathcal{P}_2(M, \mathsf{d})$ be a given probability measure. The measure m is said to satisfy a Talagrand inequality (or a transportation cost inequality) with constant K iff for all $\nu \in \mathcal{P}_2(M, \mathsf{d})$

$$\mathsf{d}_W(\nu, m) \le \sqrt{\frac{2\ \mathrm{Ent}(\nu|m)}{K}}. \tag{1.5.1}$$

Such an inequality was first proved by Talagrand in [Ta96] for the canonical Gaussian measure on $\mathbb{R}^n$. A positive rough curvature bound allows us to obtain a weaker inequality, in terms of the perturbation d_W^{+h} of the Wasserstein distance:

Proposition 1.5.1. ("h-Talagrand Inequality"). *Assume that (M, d, m) is a metric measure space which has h-$\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \ge K$ for some numbers $h > 0$ and $K > 0$. Then for each $\nu \in \mathcal{P}_2(M, \mathsf{d})$ we have*

$$\mathsf{d}_W^{+h}(\nu, m) \le \sqrt{\frac{2\ \mathrm{Ent}(\nu|m)}{K}}. \tag{1.5.2}$$

We will call (1.5.2) h-Talagrand inequality.

Proof. Since we assumed that m is a probability measure, for any $\nu \in \mathcal{P}_2(M, \mathsf{d})$ the entropy functional is nonnegative: $\mathrm{Ent}(\nu|m) \geq -\log m(M) = 0$, by Jensen's inequality. The curvature bound h-$\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \geq K$ implies that for the pair of measures ν and m and for each $t \in [0,1]$ there exists an h-rough t-intermediate point $\eta_t \in \mathcal{P}_2(M, \mathsf{d})$ such that

$$\mathrm{Ent}(\eta_t|m) \leq (1-t)\mathrm{Ent}(\nu|m) - \frac{K}{2}t(1-t)\,\mathsf{d}_W^{+h}(\nu, m)^2. \tag{1.5.3}$$

If $\mathrm{Ent}(\nu|m) < \frac{K}{2}\,\mathsf{d}_W^{+h}(\nu, m)^2$ then there exists an $\epsilon > 0$ such that $\mathrm{Ent}(\nu|m) + \epsilon < \frac{K}{2}\,\mathsf{d}_W^{+h}(\nu, m)^2$. This together with (1.5.3) would imply

$$\mathrm{Ent}(\eta_t|m) < \frac{K}{2}(1-t)^2\,\mathsf{d}_W^{+h}(\nu, m)^2 - \epsilon(1-t)$$

for each $t \in [0,1]$. We choose now t very close to 1, such that $0 < 1-t < \epsilon$ and $K(1-t)^2\,\mathsf{d}_W^{+h}(\nu, m)^2 < \epsilon^2$. This entails $\mathrm{Ent}(\eta_t|m) < -\epsilon^2/2 < 0$, in contradiction with the fact that the entropy functional is nonnegative. Therefore $\mathrm{Ent}(\nu|m) \geq \frac{K}{2}\,\mathsf{d}_W^{+h}(\nu, m)^2$, which is precisely our claim. $\square$

A Talagrand inequality for the measure m implies a concentration of measure inequality for m (see for instance [Ma97]).

For a given Borel set $A \subset M$ denote the (open) r-neighborhood of A by $B_r(A) := \{x \in M : \mathsf{d}(x, A) < r\}$ for $r > 0$. The concentration function of (M, d, m) is defined as

$$\alpha_{(M,\mathsf{d},m)}(r) := \sup\left\{1 - m(B_r(A)) : A \in \mathcal{B}(M), m(A) \geq \frac{1}{2}\right\}, \quad r > 0.$$

We refer to [Le01] for further details on measure concentration.

The following result shows that positive rough curvature bound implies a normal concentration inequality, via h-Talagrand inequality.

Proposition 1.5.2. *Let (M, d, m) be a metric measure space with h-$\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m)$ $\geq K > 0$ for some $h > 0$. Then there exists an $r_0 > 0$ such that for all $r \geq r_0$*

$$\alpha_{(M,\mathsf{d},m)}(r) \leq e^{-Kr^2/8}.$$

Proof. We follow essentially the argument of K. Marton used in [Ma97] for obtaining concentration of measure out of a Talagrand inequality for the Wasserstein distance of order 1. Let $A,\ B \in \mathcal{B}(M)$ be given with $m(A), m(B) > 0$. Consider the conditional probabilities $m_A = m(\cdot|A)$ and $m_B = m(\cdot|B)$. For these measures the h-Talagrand inequality holds:

$$\mathsf{d}_W^{+h}(m_A, m) \le \sqrt{\frac{2\ \mathrm{Ent}(m_A|m)}{K}}, \quad \mathsf{d}_W^{+h}(m_B, m) \le \sqrt{\frac{2\ \mathrm{Ent}(m_B|m)}{K}}. \tag{1.5.4}$$

Let q_A and q_B be the $+h$-optimal couplings of m_A, m and m_B, m respectively. According to [Du89], section 11.8, there exists a probability measure $\hat{q}$ on $M \times M \times M$ such that its projection on the first two factors is q_A and the projection on the last two factors is q_B. Then we have in turn

$$\begin{aligned}
\mathsf{d}_W^{+h}(m_A, m) \quad + \quad & \mathsf{d}_W^{+h}(m, m_B) = \left\{ \int_{M\times M\times M} \left[(\,\mathsf{d}(x_1,x_2) - h)_+ \right]^2 d\hat{q}(x_1,x_2,x_2) \right\}^{1/2} \\
+ \quad & \left\{ \int_{M\times M\times M} \left[(\,\mathsf{d}(x_2,x_3) - h)_+ \right]^2 d\hat{q}(x_1,x_2,x_2) \right\}^{1/2} \\
\ge \quad & \left\{ \int_{M\times M\times M} \left[(\,\mathsf{d}(x_1,x_2) - h)_+ + (\,\mathsf{d}(x_2,x_3) - h)_+ \right]^2 d\hat{q}(x_1,x_2,x_2) \right\}^{1/2} \\
\ge \quad & \left\{ \int_{M\times M\times M} \left[(\,\mathsf{d}(x_1,x_2) + \mathsf{d}(x_2,x_3) - 2h)_+ \right]^2 d\hat{q}(x_1,x_2,x_2) \right\}^{1/2} \\
\ge \quad & \left\{ \int_{M\times M\times M} \left[(\,\mathsf{d}(x_1,x_3) - 2h)_+ \right]^2 d\hat{q}(x_1,x_2,x_2) \right\}^{1/2}.
\end{aligned}$$

Assume now that $\mathsf{d}(A,B) \ge 2h$. Since the projection on the first factor of $\hat{q}$ is m_A and the projection on the last factor is m_B, the support of $\hat{q}$ must be a subset of $A \times M \times B$, hence

$$\left\{ \int_{M\times M\times M} \left[(\,\mathsf{d}(x_1,x_3) - 2h)_+ \right]^2 d\hat{q}(x_1,x_2,x_2) \right\}^{1/2} \ge \mathsf{d}(A,B) - 2h.$$

The above estimates together with (1.5.4) imply

$$\begin{aligned}
\mathsf{d}(A,B) - 2h \quad \le & \quad \sqrt{\frac{2\ \mathrm{Ent}(m_A|m)}{K}} + \sqrt{\frac{2\ \mathrm{Ent}(m_B|m)}{K}} \\
= & \quad \sqrt{\frac{2}{K} \log \frac{1}{m(A)}} + \sqrt{\frac{2}{K} \log \frac{1}{m(B)}}\ .
\end{aligned}$$

If we choose now $2h \le r$ and for a given $A \in \mathcal{B}(M)$ we replace B by $\complement B_r(A)$, we get

$$r - 2h \le \sqrt{\frac{2}{K} \log \frac{1}{m(A)}} + \sqrt{\frac{2}{K} \log \frac{1}{1 - m(B_r(A))}}\ .$$

Hence, for $m(A) \geq \frac{1}{2}$

$$r - 2h \leq \sqrt{\frac{2}{K} \log 2} + \sqrt{\frac{2}{K} \log \frac{1}{1 - m(B_r(A))}} \ .$$

Therefore whenever $r \geq 2\sqrt{\frac{2}{K} \log 2} + 4h$ for instance we have

$$\frac{r}{2} \leq \sqrt{\frac{2}{K} \log \frac{1}{1 - m(B_r(A))}} \ ,$$

or equivalently

$$1 - m(B_r(A)) \leq e^{-Kr^2/8},$$

which ends the proof.

□

In [BG99] it has been shown that a Talagrand type inequality implies exponential integrability of the Lipshitz functions. We prove further that an h-Talagrand inequality leads to the same conclusion.

Theorem 1.5.3. *Assume that (M, d) is a metric space and let $h > 0$ be given. If m is a probability measure on (M, d) that satisfies an h-Talagrand inequality of constant $K > 0$ then all Lipschitz functions are exponentially integrable. More precisely, for any Lipschitz function φ with $\|\varphi\|_{\mathrm{Lip}} \leq 1$ and $\int \varphi \, dm = 0$ we have*

$$\forall t > 0 \qquad \int_M e^{t\varphi} dm \leq e^{\frac{t^2}{2K} + ht}, \tag{1.5.5}$$

or equivalently, for any Lipschitz function φ

$$\forall t > 0 \quad \int_M e^{t\varphi} dm \leq \exp\left(t \int_M \varphi \ dm \right) \exp\left(\frac{t^2}{2K} \|\varphi\|^2_{\mathrm{Lip}} + ht\|\varphi\|_{\mathrm{Lip}} \right). \tag{1.5.6}$$

Proof. The proof we present here extends the one given in [BG99]. Let f be a probability density with $f \log f$ integrable with respect to m. The h-Talagrand inequality implies

$$\mathsf{d}_W^{+h}(fm, m) \leq \sqrt{\frac{2}{K} \int_M f \log f \ dm} \leq \frac{t}{2K} + \frac{1}{t} \int_M f \log f \ dm$$

for each $t > 0$. We consider now the Wasserstein distance of order 1 of two probability measures μ and ν

$$\mathsf{d}_W^1(\mu, \nu) := \inf \int_{M \times M} \mathsf{d}(x_0, x_1) \ dq(x_0, x_1),$$

where q ranges over all couplings of μ and ν. If $\widetilde{q}$ is a $+h$-optimal coupling of fm and m then by the Cauchy-Schwartz inequality,

$$\begin{aligned} \mathsf{d}_W^{+h}(fm,m) &= \left\{\int_{M\times M}\left[(\,\mathsf{d}(x_0,x_1)-h)_+\right]^2 d\widetilde{q}(x_0,x_1)\right\}^{1/2} \\ &\geq \int_{M\times M}(\,\mathsf{d}(x_0,x_1)-h)_+\, d\widetilde{q}(x_0,x_1) \geq \mathsf{d}_W^1(fm,m)-h. \end{aligned}$$

The Kantorovich-Rubinstein theorem gives the following duality formula

$$\mathsf{d}_W^1(fm,m) = \sup_{\|\varphi\|_{\mathrm{Lip}}\leq 1}\left\{\int_M \varphi f\ dm - \int_M \varphi\ dm\right\}.$$

If φ is a Lipschitz function that satisfies the assumptions of the theorem ($\|\varphi\|_{\mathrm{Lip}}\leq 1$ and $\int \varphi\ dm = 0$) then

$$\int_M \varphi f\ dm \leq \mathsf{d}_W^{+h}(fm,m) + h \leq \frac{t}{2K} + \frac{1}{t}\int_M f\log f\ dm + h,$$

which can be written as

$$\int_M \left(t\varphi - \frac{t^2}{2K}\right) f\ dm \leq \int_M f\log f\ dm + ht. \tag{1.5.7}$$

This estimate should take place for any probability density f. Therefore one can take

$$f = e^{t\varphi-\frac{t^2}{2K}}\left(\int_M e^{t\varphi-\frac{t^2}{2K}}dm\right)^{-1}$$

in formula (1.5.7) and obtain

$$\begin{aligned} \left\{\int_M \left(t\varphi-\frac{t^2}{2K}\right)e^{t\varphi-\frac{t^2}{2K}}dm\right\}\left(\int_M e^{t\varphi-\frac{t^2}{2K}}dm\right)^{-1} &\leq \int_M e^{t\varphi-\frac{t^2}{2K}}\left(\int_M e^{t\varphi-\frac{t^2}{2K}}dm\right)^{-1} \\ &\quad\cdot\left\{t\varphi-\frac{t^2}{2K}-\log\left(\int_M e^{t\varphi-\frac{t^2}{2K}}dm\right)\right\}dm + ht. \end{aligned}$$

This yields

$$\log\left(\int_M e^{t\varphi-\frac{t^2}{2K}}dm\right)dm \leq ht,$$

that proves the claim (1.5.5). The general estimate (1.5.6) is a consequence of (1.5.5) applied to the function $\psi = \frac{1}{\|\varphi\|_{\mathrm{Lip}}}\left[\varphi - \int\varphi\ dm\right]$. □

Chapter 2

The rough curvature-dimension condition for metric measure spaces

We shall introduce in the sequel a stronger condition then the rough lower curvature bound. The examples we studied in the previous chapter were discrete analogues of finite dimensional Riemannian manifolds. They have intuitively not only a "curvature" along with the manifold, but also a certain "dimensional" aspect. A planar graph has intuitively dimension 2, since it can be drawn in the plane. We aim to capture this dimensional constriction into a curvature-dimension type condition, in order to obtain more geometrical consequences.

We define and study a rough curvature-dimension condition h-$\mathsf{CD}(K, N)$ for metric measure spaces, where K plays the role of the lower curvature bound and N the role of the upper bound for the dimension. We prove that a (continuous) metric measure space (M, d, m) which can be approximated by a family $\{(M_h, \mathsf{d}_h, m_h)\}_{h>0}$ of (discrete) metric measure spaces which satisfy a rough curvature condition h-$\mathsf{CD}(K, N)$, with the mesh size h going to zero, satisfies a curvature-dimension condition $\mathsf{CD}(K, N)$. We show also that the rough curvature dimension condition can be preserved under the converse procedure: a discretization of a metric measure space with $\mathsf{CD}(K, N)$ property satisfies an h-$\mathsf{CD}(K, N)$ condition is the mesh of the discretization is small enough. We prove a generalization of Brunn-Minkowski inequality and a Bonnet-Myers type theorem.

2.1 Preliminaries

We start again with a metric measure space (M, d, m), where (M, d) is a complete and separable metric space and m is a locally finite measure on the Borel σ-algebra $\mathcal{B}(M)$ of M.

A point z in M is called a *t-intermediate point* between x and y for some $t \in [0,1]$ if $\mathsf{d}(x,z) = t \cdot \mathsf{d}(x,y)$ and $\mathsf{d}(z,y) = (1-t) \cdot \mathsf{d}(x,y)$.

Instead of the relative entropy $\mathrm{Ent}(\cdot|m)$ we will use the Rényi entropy functional, which depends also on a parameter $N \geq 1$ that will play the role of the dimension in the following material. The *Rényi entropy functional* with respect to our reference measure m is defined as

$$S_N(\cdot|m) : \mathcal{P}_2(M, \mathsf{d}) \to \mathbb{R}$$

with

$$S_N(\nu|m) := -\int_M \rho^{-1/N} d\nu,$$

where ρ is the density of the absolutely continuous part ν^c with respect to m in the Lebesgue decomposition $\nu = \nu^c + \nu^s = \rho m + \nu^s$ of the measure $\nu \in \mathcal{P}_2(M, \mathsf{d})$.

Lemma 1.1 from [St06b] states that

Lemma 2.1.1. *Assume that $m(M)$ is finite.*

(i) For each $N > 1$ the Rényi entropy functional $S_N(\cdot|m)$ is lower semicontinuous and satisfies

$$-m(M)^{1/N} \leq S_N(\cdot|m) \leq 0 \text{ on } \mathcal{P}_2(M, \mathsf{d}).$$

(ii) For any $\nu \in \mathcal{P}_2(M, \mathsf{d})$

$$\mathrm{Ent}(\cdot|m) = \lim_{N\to\infty} N(1 + S_N(\nu|m)).$$

For given $K, N \in \mathbb{R}$ with $N \geq 1$ and $(t,\theta) \in [0,1] \times \mathbb{R}_+$ we use the notation

$$\tau_{K,N}^{(t)}(\theta) = \begin{cases} \infty\,, & \text{if } K\theta^2 \geq (N-1)\pi^2 \\ t^{\frac{1}{N}} \left(\sin\left(\sqrt{\frac{K}{N-1}} t\theta\right) \Big/ \sin\left(\sqrt{\frac{K}{N-1}}\theta\right)\right)^{1-\frac{1}{N}}, & \text{if } 0 < K\theta^2 < (N-1)\pi^2 \\ t\,, & \text{if } K\theta^2 = 0 \text{ or} \\ & \text{if } K\theta^2 < 0 \text{ and } N = 1 \\ t^{\frac{1}{N}} \left(\sinh\left(\sqrt{\frac{-K}{N-1}} t\theta\right) \Big/ \sinh\left(\sqrt{\frac{-K}{N-1}}\theta\right)\right)^{1-\frac{1}{N}}, & \text{if } K\theta^2 < 0 \text{ and } N > 1. \end{cases}$$

Remark 2.1.2. For arbitrarily fixed $t \in (0,1)$ and $\theta \in (0,\infty)$ the function $(K,N) \to \tau_{K,N}^{(t)}(\theta)$ is continuous, nondecreasing in K and non-increasing in N.

The curvature-dimension condition for geodesic spaces (M, d, m) was introduced in [St06b] in the following way:

Definition 2.1.3. Given two numbers $K,\,N\in\mathbb{R}$ with $N\geq 1$ we say that a metric measure space $(M,\,\mathsf{d},m)$ satisfies the *curvature-dimension condition* $\mathsf{CD}(K,N)$ iff for each pair $\nu_0,\,\nu_1\in\mathcal{P}_2(M,\,\mathsf{d},m)$ there exist an optimal coupling q of $\nu_0,\,\nu_1$ and a geodesic $\Gamma:[0,1]\to\mathcal{P}_2(M,\,\mathsf{d},m)$ connecting $\nu_0,\,\nu_1$ and with

$$\begin{aligned} S_{N'}(\eta_t|m) \;\leq\; -\int \Big[\tau^{(1-t)}_{K,N'}(\,\mathsf{d}(x_0,x_1))\cdot\rho_0^{-1/N'}(x_0) \\ +\,\tau^{(t)}_{K,N'}(\,\mathsf{d}(x_0,x_1))\cdot\rho_1^{-1/N'}(x_1)\Big]\,dq(x_0,x_1) \end{aligned} \tag{2.1.1}$$

for all $t\in[0,1]$ and all $N'\geq N$. Here ρ_i denotes the density functions of the absolutely continuous parts of ν_i with respect to m, $i=1,2$.

If $(M,\,\mathsf{d},m)$ has finite mass and satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$ for some K and N then it has curvature $\geq K$ in the sense of Definition 1.1.2. In other words, the condition $\mathbb{C}\mathrm{urv}(M,\,\mathsf{d},m)\geq K$ may be interpreted as the curvature-dimension condition $\mathsf{CD}(K,\infty)$ for the space $(M,\,\mathsf{d},m)$.

For Riemannian manifolds the curvature-dimension condition $\mathsf{CD}(K,N)$ reverts to "Ricci curvature bounded below by K and dimension bounded above by N", as it is shown in Theorem 1.7 from [St06b].

Theorem 2.1.4. *Let M be a complete Riemannian manifold with Riemannian distance d and Riemannian volume m and let numbers $K,\,N\in\mathbb{R}$ with $N\geq 1$ be given.*

(i) The metric measure space $(M,\,\mathsf{d},m)$ satisfies the curvature-dimension condition $\mathsf{CD}(K,N)$ if and only if the Riemannian manifold has Ricci curvature $\geq K$ and dimension $\leq N$.

(ii) Moreover, in this case for every measurable function $V:M\to\mathbb{R}$ the weighted space $(M,\,\mathsf{d},V\cdot m)$ satisfies the curvature-dimension condition $\mathsf{CD}(K+K',N+N')$ provided

$$\mathrm{Hess}\;V^{1/N'}\leq-\frac{K'}{N'}\cdot V^{1/N'}$$

for some numbers $K'\in\mathbb{R}$, $N'>0$ in the sense that

$$V(\gamma_t)^{1/N'}\geq\sigma^{(1-t)}_{K',N'}(\,\mathsf{d}(\gamma_0,\gamma_1))V(\gamma_0)^{1/N'}+\sigma^{(t)}_{K',N'}(\,\mathsf{d}(\gamma_0,\gamma_1))V(\gamma_1)^{1/N'}$$

for each geodesic $\gamma:[0,1]\to M$ and each $t\in[0,1]$. Here $\sigma^{(t)}_{K,N}(\theta):= sin\left(\sqrt{\frac{K}{N}}t\theta\right)\Big/ sin\left(\sqrt{\frac{K}{N}}\theta\right)$ if $0<K\theta^2<N\pi^2$ and with appropriate modifications otherwise.

2.2 The Rough Curvature-Dimension Condition

We will introduce in the sequel the rough curvature-dimension condition and give some basic properties of it. There are various ways to extend the Definition 2.1.3 to make it applicable to more general spaces than geodesic spaces. It matters where and how we plug in our "h". There are two ways that seem more natural, each of them with its advantages. For the moment we recall and refine the definition of the h-rough t-intermediate point of two given points:

Definition 2.2.1. (i) If (M, d) is a metric space and $h \geq 0$, $t \in [0,1]$ are given real numbers we say that x_t is an *h-rough t-intermediate point* of x_0 and x_1 in M if

$$\begin{cases} \mathsf{d}(x_0, x_t) & \leq & t\,\mathsf{d}(x_0, x_1) + h \\ \mathsf{d}(x_t, x_1) & \leq & (1-t)\,\mathsf{d}(x_0, x_1) + h \end{cases}$$

(ii) We say that x_t is an *h-rough t-intermediate point* of x_0 and x_1 *in the strong sense* if

$$(1-t)\,\mathsf{d}(x_0, x_t)^2 + t\,\mathsf{d}(x_t, x_1)^2 \leq t(1-t)\,\mathsf{d}(x_0, x_1)^2 + h^2. \tag{2.2.1}$$

Remark 2.2.2. If x_t is an h-rough t-intermediate point of x_0 and x_1 in the strong sense then x_t is an h-rough t-intermediate point of x_0 and x_1. Indeed, the triangle inequality $|\,\mathsf{d}(x_0, x_1) - \mathsf{d}(x_0, x_t)| \leq \mathsf{d}(x_t, x_1)$ together with (2.2.1) yield

$$(1-t)\,\mathsf{d}(x_0, x_t)^2 + t|\,\mathsf{d}(x_0, x_1) - \mathsf{d}(x_0, x_t)|^2 \leq t(1-t)\,\mathsf{d}(x_0, x_1)^2 + h^2$$

or equivalently

$$(1-t)\,\mathsf{d}(x_0, x_t)^2 + t\,\mathsf{d}(x_0, x_1)^2 + t\,\mathsf{d}(x_0, x_t)^2 - 2t\,\mathsf{d}(x_0, x_t)\,\mathsf{d}(x_0, x_1) \leq t(1-t)\,\mathsf{d}(x_0, x_1)^2 + h^2,$$

which gives

$$\mathsf{d}(x_0, x_t)^2 - 2t\,\mathsf{d}(x_0, x_t)\,\mathsf{d}(x_0, x_1) \leq -t^2\,\mathsf{d}(x_0, x_1)^2 + h^2 \;\Leftrightarrow\; [\,\mathsf{d}(x_0, x_t) - t\,\mathsf{d}(x_0, x_1)]^2 \leq h^2.$$

Similarly one obtains the analogous inequality corresponding to $\mathsf{d}(x_t, x_1)$.

Remark 2.2.3. With the additional assumption that M has finite diameter L, weak h-rough t-intermediate points are strong h' rough t-intermediate points for $h' = (2Lh)^{1/2}$.

We get in this way two possible definitions for a rough curvature-dimension condition.

Definition 2.2.4. (i) Given three numbers $K, N, h \in \mathbb{R}$ with $N \geq 1$ and $h \geq 0$ we say that a metric measure space (M, d, m) satisfies the *rough curvature-dimension condition* h-$\mathsf{CD}(K, N)$ iff for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ there exists a δh-optimal coupling q of ν_0, ν_1 such that for any $t \in [0, 1]$ there exists an h-rough t-intermediate point $\eta_t \in \mathcal{P}_2(M, \mathsf{d}, m)$ of ν_0, ν_1 with

$$\begin{aligned} S_{N'}(\eta_t|m) \quad \leq \quad -\int \Big[& \tau^{(1-t)}_{K,N'}((\, \mathsf{d}(x_0, x_1) - \delta h)_+) \cdot \rho_0^{-1/N'}(x_0) \\ & + \tau^{(t)}_{K,N'}((\, \mathsf{d}(x_0, x_1) - \delta h)_+) \cdot \rho_1^{-1/N'}(x_1) \Big] \, dq(x_0, x_1) \end{aligned} \tag{2.2.2}$$

for all $N' \geq N$. Here ρ_i denotes the density of the absolutely continuous part of ν_i w.r.t. m, $i = 0, 1$, and $\delta = -1$ for $K < 0$ and $\delta = 1$ for $K \geq 0$, where $(\cdot)_+$ denotes the positive part.

(ii) We say that (M, d, m) satisfies the *rough curvature-dimension condition in the strong sense* h-$\mathsf{CD}^s(K, N)$ if for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ there exists a δh-optimal coupling q of ν_0, ν_1 such that for any $t \in [0, 1]$ there exists an h-rough t-intermediate point $\eta_t \in \mathcal{P}_2(M, \mathsf{d}, m)$ of ν_0, ν_1 in the strong sense satisfying (2.2.2) for all $N' \geq N$.

As we will see, the first definition is better suited for stability under discretizations whereas the second is more powerful for obtaining geometrical consequences.

Remark 2.2.5. According Remark 2.2.3 on bounded spaces the rough curvature-dimension condition and the strong rough curvature-dimension condition are equivalent, modulo changes of the coarseness parameter h.

Remark 2.2.6. For $K = 0$ inequality (2.2.2) reads

$$S_{N'}(\eta_t|m) \leq (1 - t) \cdot S_{N'}(\nu_0|m) + t \cdot S_{N'}(\nu_1|m),$$

so the rough curvature-dimension condition h-$\mathsf{CD}(0, N)$ requires the Rényi entropy functionals $S_{N'}(\cdot|m)$ to be weakly convex on $\mathcal{P}_2(M, \mathsf{d}, m)$ along "h-rough geodesics" for all $N' \geq N$.

Proposition 2.2.7. *Suppose that (M, d, m) is a metric measure space that satisfies the h-$\mathsf{CD}(K, N)$ condition for some numbers $h \geq 0$, $K, N \in \mathbb{R}$. Then the following properties hold:*

(i) (M, d, m) satisfies also the h-$\mathsf{CD}(K', N')$ condition for any $K' \leq K$ and $N' \geq N$. If $K \leq 0$ then (M, d, m) satisfies also the h'-$\mathsf{CD}(K, N)$ condition for any $h' \geq h$.

(ii) Any metric space (M', d', m') that is isomorphic to (M, d, m) satisfies the same h-$\mathsf{CD}(K, N)$ condition.

(iii) *For any $\alpha, \beta > 0$ the metric measure space $(M, \alpha\,\mathsf{d}, \beta m)$ satisfies the αh-$\mathsf{CD}(\alpha^{-2}K, N)$ condition.*

(iv) *If (M, d, m) has finite mass then h-$\mathbb{C}\mathrm{urv}(M, \mathsf{d}, m) \geq K$.*

Proof. *(i), (ii)* Obvious.

(iii) Consider $\nu_0, \nu_1 \in \mathcal{P}_2(M, \alpha\,\mathsf{d}, \beta m)$. Then $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ and denote by ρ_i the density of ν_i with respect to m, for $i = 0, 1$. Denote by δ the sign of K. Let q be a δh-optimal coupling and $\eta = \rho m$ an h-rough t-intermediate point of ν_0, ν_1 with respect to the metric d, satisfying condition (2.1.1) for any $N' \geq N$. Then q is a $\delta(\alpha h)$-optimal coupling and η is an αh-rough t-intermediate point of ν_0, ν_1 with respect to the metric $\alpha\,\mathsf{d}$ and we have

$$\begin{aligned}
S_{N'}(\eta|\beta m) &= -\int_M (\rho/\beta)^{1-1/N'}\, d(\beta m) = \beta^{1/N'} S_{N'}(\eta|m) \\
&\leq -\beta^{1/N'} \int \Big[\tau_{K,N'}^{(1-t)}((\,\mathsf{d}(x_0,x_1) - \delta h)_+) \cdot \rho_0^{-1/N'}(x_0) \\
&\qquad + \tau_{K,N'}^{(t)}((\,\mathsf{d}(x_0,x_1) - \delta h)_+) \cdot \rho_1^{-1/N'}(x_0)\Big]\, dq(x_0,x_1) \\
&= \int \Big[\tau_{\alpha^{-2}K,N'}^{(1-t)}((\alpha\,\mathsf{d}(x_0,x_1) - \delta(\alpha h))_+)(\rho_0/\beta)^{-1/N'}(x_0) \\
&\qquad + \tau_{\alpha^{-2}K,N'}^{(t)}((\alpha\,\mathsf{d}(x_0,x_1) - \delta(\alpha h))_+)(\rho_1/\beta)^{-1/N'}(x_0)\Big]\, dq(x_0,x_1)
\end{aligned}$$

for any $N' \geq N$, which gives the αh-$\mathsf{CD}(\alpha^{-2}K, N)$ condition for the metric measure space $(M, \alpha\,\mathsf{d}, \beta m)$.

(iv) In order to prove the h-curvature bound in the sense of the Definition 1.2.7 we consider $\nu_0, \nu_1 \in \mathcal{P}_2^*(M, \mathsf{d}, m)$. Since the space (M, d, m) satisfies the h-$\mathsf{CD}(K, N)$ condition, one can find a δh-optimal coupling q and for any $t \in [0,1]$ there is an h-rough t-intermediate point $\eta_t \in \mathcal{P}_2(M, \mathsf{d}, m)$ of ν_0 and ν_1 that fulfill condition (2.1.1) for any $N' \geq N$. With our assumption $m(M) < \infty$, Lemma 2.1.1 gives us the relative entropy of η_t with respect to m as $\mathrm{Ent}(\eta_t|m) = \lim_{N'\to\infty} N'(1 + S'_N(\eta_t|m))$. Therefore,

$$\begin{aligned}
\mathrm{Ent}(\eta_t|m) &- (1-t)\mathrm{Ent}(\nu_0|m) - t\mathrm{Ent}(\nu_1|m) \\
&= \lim_{N'\to\infty} N'(S'_N(\eta_t|m) - (1-t)S'_N(\nu_0|m) - tS'_N(\nu_0|m)) \\
&\leq \lim_{N'\to\infty} \int \Big\{ N'\Big[(1-t) - \tau_{K,N'}^{(1-t)}((\,\mathsf{d}(x_0,x_1) - \delta h)_+)\Big]\rho_0^{-1/N'}(x_0) \\
&\qquad + N'\Big[t - \tau_{K,N'}^{(t)}((\,\mathsf{d}(x_0,x_1) - \delta h)_+)\Big]\rho_1^{-1/N'}(x_1)\Big\}\, dq(x_0,x_1) \\
&\leq \lim_{N'\to\infty} \int \Big\{ N'\Big[(1-t) - \tau_{K,N'}^{(1-t)}((\,\mathsf{d}(x_0,x_1) - \delta h)_+)\Big] \\
&\qquad + N'\Big[t - \tau_{K,N'}^{(t)}((\,\mathsf{d}(x_0,x_1) - \delta h)_+)\Big]\Big\}\, dq(x_0,x_1).
\end{aligned}$$

If $0 < K\theta^2 < (N-1)\pi^2$ then

$$\lim_{N'\to\infty} N'\left[t - \tau^{(t)}_{K,N'}(\theta)\right] = \lim_{N'\to\infty} N'\left[t - t^{\frac{1}{N}}\left(\frac{\sin\left(\sqrt{\frac{K}{N-1}}t\theta\right)}{\sin\left(\sqrt{\frac{K}{N-1}}\theta\right)}\right)^{1-\frac{1}{N}}\right] = \frac{K\theta^2}{6}(t^3-t).$$

We get the same limit $K\theta^2(t^3-t)/6$ for the other three interpretations of $\tau^{(t)}_{K,N'}(\theta)$, therefore one can conclude

$$\begin{aligned}\mathrm{Ent}(\eta_t|m) &\leq \int \frac{K\left(\,\mathsf{d}(x_0,x_1)-\delta h)_+\right)^2}{6}\{[(1-t)^3-(1-t)]+(t^3-t)\}dq(x_0,x_1)\\ &= -\frac{K}{2}t(1-t)\int\left[(\,\mathsf{d}(x_0,x_1)-\delta h)_+\right]^2 dq(x_0,x_1) = \,\mathsf{d}_W^{\delta h}(\nu_0,\nu_1)^2.\end{aligned}$$

□

Remark 2.2.8. The Proposition 2.2.7 holds true of we replace everywhere h-$\mathsf{CD}(K,N)$ by h-$\mathsf{CD}^s(K,N)$.

Remark 2.2.9. The item (iv) of Proposition 2.2.7 shows that for a metric measure space of finite mass the condition h-$\mathbb{C}\mathrm{urv}(M,\mathsf{d},m)\geq K$ may be seen as a rough curvature-dimension condition h-$\mathsf{CD}(K,\infty)$.

2.3 Geometrical consequences of the rough curvature-dimension condition

In the case of the geodesic spaces, for each geodesic Γ from the Wasserstein space the mass is transported along geodesics of the underlying space with endpoints in the supports of $\Gamma(0)$ and $\Gamma(1)$ respectively (see [St06a] Lemma 2.11). In our more general framework, for an arbitrary h-geodesic Γ in $\mathcal{P}_2(M,\mathsf{d})$ the mass is not necessarily transported along h-geodesics from M. However, the following result shows that if Γ is a strong h-geodesic then the mass is mostly transported along strong h'-geodesics from M that joint points from supp $\Gamma(0)$ and supp $\Gamma(1)$ and with $h' > h$ sufficiently small.

Lemma 2.3.1. *Let (M,d,m) be a metric measure space and μ_0, μ_1 two probability measures on it and denote $A_i := \mathrm{supp}[\mu_i]$, $i=0,1$. Assume that η is an h-rough*

t-intermediate point in the strong sense of μ_0 and μ_1 in $\mathcal{P}_2(M, \mathsf{d}, m)$, for some numbers $h \geq 0$ and $t \in [0,1]$. For $\lambda \geq 0$ we denote

$$\begin{aligned} A_t^\lambda := \{y \in M : \exists (x_0,x_1) \in A_0 \times A_1 \quad : \quad & (1-t)\,\mathsf{d}(x_0,y)^2 + t\,\mathsf{d}(y,x_1)^2 \\ & \leq t(1-t)(\,\mathsf{d}(x_0,x_1)^2 + \lambda^2)\}. \end{aligned}$$

Then the following estimate holds:

$$\eta(\complement A_t^\lambda) \leq h^2/\lambda^2 \text{ for any } \lambda > 0. \tag{2.3.1}$$

Moreover, if $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots$ then

$$\sum_{i=1}^\infty \lambda_i^2 \cdot \eta(A_t^{\lambda_{i+1}} \setminus A_t^{\lambda_i}) \leq h^2$$

or, equivalently,

$$\sum_{i=1}^\infty \eta(\complement A_t^{\lambda_i})(\lambda_i^2 - \lambda_{i-1}^2) \leq h^2.$$

Proof. Let q_0 be an optimal coupling of μ_0 and η, and q_1 be an optimal coupling of η and μ_1. One can construct then a probability measure $\widehat{q}$ on $M \times M \times M$ such that the projection on the first two factors is q_0 and the projection on the last two factors is q_1 (cf. [Du89], section 11.8). Therefore,

$$\mathsf{d}_W(\mu_0,\eta)^2 = \int_{M^3} \mathsf{d}(x_0,y)^2 d\widehat{q}(x_0,y,x_1), \;\; \mathsf{d}_W(\eta,\mu_1)^2 = \int_{M^3} \mathsf{d}(y,x_1)^2 d\widehat{q}(x_0,y,x_1).$$

Because the inequality

$$t(1-t)\,\mathsf{d}(x_0,x_1)^2 \leq (1-t)\,\mathsf{d}(x_0,y)^2 + t\,\mathsf{d}(y,x_1)^2$$

always holds, for $\lambda > 0$ we have

$$\begin{aligned} \eta(\complement A_t^\lambda) &= \widehat{q}(A_0 \times \complement A_t^\lambda \times A_1) \\ &\leq \frac{1}{\lambda^2} \int_{A_0 \times \complement A_t^\lambda \times A_1} \big[(1-t)\,\mathsf{d}(x_0,y)^2 + t\,\mathsf{d}(y,x_1)^2 \\ &\qquad\qquad -t(1-t)\,\mathsf{d}(x_0,x_1)^2\big]\, d\widehat{q}(x_0,y,x_1) \\ &\leq \frac{1}{\lambda^2} \int_{M^3} \big[(1-t)\,\mathsf{d}(x_0,y)^2 + t\,\mathsf{d}(y,x_1)^2 \\ &\qquad\qquad -t(1-t)\,\mathsf{d}(x_0,x_1)^2\big]\, d\widehat{q}(x_0,y,x_1) \\ &\leq \frac{1}{\lambda^2} \big[(1-t)\,\mathsf{d}_W(\mu_0,\eta)^2 + t\,\mathsf{d}_W(\eta,\mu_1)^2 - t(1-t)\,\mathsf{d}_W(\mu_0,\mu_1)^2\big] \\ &\leq \frac{h^2}{\lambda^2}, \end{aligned}$$

which proves the first part of the lemma.

Consider now a nondecreasing sequence $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots$. Since $M = A_t^0 \dot\cup \left(\dot\bigcup_{i=0}^{\infty} \left(A_t^{\lambda_{i+1}} - A_t^{\lambda_i} \right) \right)$ we have in turn

$$\begin{aligned}
&t(1-t)\,\mathsf{d}_W(\mu_0,\mu_1)^2 + h^2 \geq (1-t)\,\mathsf{d}_W(\mu_0,\eta)^2 + t\,\mathsf{d}_W(\eta,\mu_1)^2 \\
&= \int_{M^3} \left[(1-t)\,\mathsf{d}(x_0,y)^2 + t\,\mathsf{d}(y,x_1)^2\right] d\widehat{q}(x_0,y,x_1) \\
&= \int_{A_0\times A_t^0\times A_1} \left[(1-t)\,\mathsf{d}(x_0,y)^2 + t\,\mathsf{d}(y,x_1)^2\right] d\widehat{q}(x_0,y,x_1) \\
&+ \sum_{i=0}^{\infty} \int_{A_0\times(A_t^{\lambda_{i+1}}-A_t^{\lambda_i})\times A_1} \left[(1-t)\,\mathsf{d}(x_0,y)^2 + t\,\mathsf{d}(y,x_1)^2\right] d\widehat{q}(x_0,y,x_1) \\
&\geq t(1-t) \int_{A_0\times A_t^0\times A_1} \mathsf{d}(x_0,x_1)^2 d\widehat{q}(x_0,y,x_1) \\
&+ \sum_{i=0}^{\infty} \int_{A_0\times(A_t^{\lambda_{i+1}}-A_t^{\lambda_i})\times A_1} \left[t(1-t)\,\mathsf{d}(x_0,x_1)^2 + \lambda_i^2\right] d\widehat{q}(x_0,y,x_1) \\
&= t(1-t) \int_{M^3} \mathsf{d}(x_0,x_1)^2 d\widehat{q}(x_0,y,x_1) + \sum_{i=1}^{\infty} \lambda_i^2 \cdot \eta(A_t^{\lambda_{i+1}} \setminus A_t^{\lambda_i})
\end{aligned}$$

This leads to $\sum_{i=1}^{\infty} \lambda_i^2 \cdot \eta(A_t^{\lambda_{i+1}} \setminus A_t^{\lambda_i}) \leq h^2$. □

Having the above description of the strong h-geodesics in the Wasserstein space we shall establish a rough Brunn-Minkowski inequality for metric measure spaces that satisfy a rough curvature-dimension condition in the strong sense.

The classical Brunn-Minkowski inequality in $\mathbb{R}^n$ states that for all bounded Borel measurable subsets A and B in $\mathbb{R}^n$,

$$\mathrm{vol}_n(A+B)^{1/n} \geq \mathrm{vol}_n(A)^{1/n} + \mathrm{vol}_n(B)^{1/n}, \tag{2.3.2}$$

where $A+B := \{x+y : x \in A, y \in B\}$ is the Minkowski sum of A and B and where $\mathrm{vol}_n(\cdot)$ denotes the volume element in $\mathbb{R}^n$. Inequality (2.3.2) can be equivalently rewritten as

$$\mathrm{vol}_n\left(\frac{A+B}{2}\right)^{1/n} \geq \frac{1}{2}\mathrm{vol}_n(A)^{1/n} + \frac{1}{2}\mathrm{vol}_n(B)^{1/n},$$

in terms of the set $(A+B)/2$ of midpoints of pairs of points from A and B respectively, or even more generally as

$$\mathrm{vol}_n(tA+(1-t)B)^{1/n} \geq t\ \mathrm{vol}_n(A)^{1/n} + (1-t)\mathrm{vol}_n(B)^{1/n}$$

for any $t \in [0,1]$.

The next result extends the Brunn-Minkowski inequality to the frame of metric measure spaces satisfying an h-$\mathsf{CD}^s(K,N)$.

Proposition 2.3.2. *Let (M, d, m) be a metric measure space that has finite mass and satisfies h-$\mathsf{CD}^s(K,N)$ for some numbers $h \geq 0$, $K, N \in \mathbb{R}$, $N \geq 1$. Then for any measurable sets A_0, $A_1 \subset M$ with $m(A_0) \cdot m(A_1) > 0$, for any $t \in [0,1]$, $N' \geq N$ and any $\lambda > 0$*

$$m(A_t^\lambda)^{1/N'} + (h^2/\lambda^2)^{1-1/N'} m(\complement A_t^\lambda)^{1/N'} \geq \tau_{K,N'}^{(1-t)}(\Theta_h) m(A_0)^{1/N'} + \tau_{K,N'}^{(t)}(\Theta_h) m(A_1)^{1/N'}, \tag{2.3.3}$$

where A_t^λ is the one denoted in Lemma 2.3.1 and Θ_h is given by

$$\Theta_h := \begin{cases} \inf_{x_0 \in A_0, x_1 \in A_1} (\mathsf{d}(x_0,x_1) - h)_+, & \text{if } K \geq 0 \\ \sup_{x_0 \in A_0, x_1 \in A_1} (\mathsf{d}(x_0,x_1) + h), & \text{if } K < 0. \end{cases}$$

Corollary 2.3.3. ('Generalized Brunn-Minkowski Inequality'). *Assume that (M, d, m) is a normalized metric measure space that satisfies h-$\mathsf{CD}^s(K,N)$ for some numbers $h \geq 0$, $K, N \in \mathbb{R}$, $N \geq 1$. Then for any measurable sets A_0, $A_1 \subset M$ with $m(A_0) \cdot m(A_1) > 0$, for any $t \in [0,1]$ and $N' \geq N$*

$$m(A_t^{\sqrt{h}})^{1/N'} + h^{1-1/N'} \geq \tau_{K,N'}^{(1-t)}(\Theta_h) m(A_0)^{1/N'} + \tau_{K,N'}^{(t)}(\Theta_h) m(A_1)^{1/N'}, \tag{2.3.4}$$

with Θ_h given above.

In particular, if $K \geq 0$ then

$$m(A_t^{\sqrt{h}})^{1/N'} + h^{1-1/N'} \geq (1-t) \cdot m(A_0)^{1/N'} + t \cdot m(A_1)^{1/N'}. \tag{2.3.5}$$

Proof of the Corollary. Just take $\lambda = \sqrt{h}$ in formula (2.3.3) and use the fact that m is a probability measure. □

Proof of Proposition 2.3.2. We apply the h-$\mathsf{CD}^s(K,N)$ condition to the measures $\nu_0 := \frac{1}{m(A_0)} 1_{A_0} m$ and $\nu_1 := \frac{1}{m(A_1)} 1_{A_1} m$. Then for any $t \in [0,1]$ there exists an h-rough t-intermediate point $\eta_t \in \mathcal{P}_2(M, \mathsf{d}, m)$ in the strong sense of ν_0, ν_1 with

$$S_{N'}(\eta_t|m) \leq -\left[\tau_{K,N'}^{(1-t)}(\Theta_h) m(A_0)^{1/N'} + \tau_{K,N'}^{(t)}(\Theta_h) m(A_1)^{1/N'}\right]$$

for all $N' \geq N$. If we denote by ρ_t the density of η_t with respect to m we have then, by using Jensen and Hölder inequalities,

$$\begin{aligned}
\tau_{K,N'}^{(1-t)}(\Theta_h)m(A_0)^{1/N'} &+ \tau_{K,N'}^{(t)}(\Theta_h)m(A_1)^{1/N'} \leq \int \rho_t(y)^{1-1/N'}dm(y) \\
&= \int_{A_t^\lambda} \rho_t(y)^{1-1/N'}dm(y) + \int_{\complement A_t^\lambda} \rho_t(y)^{1-1/N'}dm(y) \\
&\leq m(A_t^\lambda)^{1/N'} + \left(\int_{\complement A_t^\lambda} \rho_t(y)dm(y)\right)^{1-1/N'} \left(\int_{\complement A_t^\lambda} dm(y)\right)^{1/N'} \\
&= m(A_t^\lambda)^{1/N'} + \eta(\complement A_t^\lambda)^{1-1/N'} m(\complement A_t^\lambda)^{1/N'} \\
&\leq m(A_t^\lambda)^{1/N'} + (h^2/\lambda^2)^{1-1/N'} m(\complement A_t^\lambda)^{1/N'},
\end{aligned}$$

where for the last inequality we have used Lemma 2.3.1. □

Remark 2.3.4. Another (stronger) discrete version of Brunn-Minkowski inequality has been introduced in [Bo07]. It has been proved there a stability result under $\mathbb{D}$-convergence and a converse result stating the stability under discretizations.

The next result provides an extension of the classical Bonnet-Myers Theorem from complete Riemannian manifolds to metric measure spaces which satisfy a rough curvature-dimension condition h-$\mathsf{CD}(K,N)$ with positive K.

Corollary 2.3.5. ('Generalized Bonnet-Myers Theorem'). *For every normalized metric measure space (M, d, m) that satisfies the rough curvature-dimension condition h-$\mathsf{CD}^s(K,N)$ for some real numbers $h > 0$, $K > 0$ and $N \geq 1$, the support of the measure m has diameter*

$$L \leq \sqrt{\frac{N-1}{K}}\pi + h.$$

In particular, if $K > 0$ and $N = 1$ then supp$[m]$ *consists of a ball of radius h.*

Proof. Suppose that x_0 and x_1 are two points in supp$[m]$ with $\mathsf{d}(x_0,x_1) \geq \sqrt{\frac{N-1}{K}}\pi + h + 4\epsilon$ and $m(B_\epsilon(x_i)) > 0$ for $i = 0,1$. Denote $A_i := B_\epsilon(x_i)$, $i = 0,1$. We can apply Corollary 2.3.3 for the sets A_0 and A_1 and for instance $t = 1/2$. According to our choice of x_0 and x_1 we have

$$\Theta_h = \inf_{x_0 \in A_0, a_1 \in A_1} (\,\mathsf{d}(x_0,x_1) - h)_+ \geq \sqrt{\frac{N-1}{K}}\pi + 2\epsilon$$

and therefore $\tau_{K,N'}^{1/2}(\Theta_h) = +\infty$, which contradicts inequality 2.3.4 in our hypothesis that m is a probability measure. □

This Bonnet-Myers type theorem comes to complete Proposition 2.2.7 (i):

Corollary 2.3.6. *Suppose that (M, d, m) is a metric measure space that satisfies the h-$\mathsf{CD}(K, N)$ condition for some numbers $h \geq 0$, $K, N \in \mathbb{R}$. Then (M, d, m) satisfies also the h'-$\mathsf{CD}(K', N')$ condition for any $h' \geq h$, $K' \leq K$ and $N' \geq N$.*

Proof. The only case that wasn't included in Proposition 2.2.7 was the one with the positive K, since in general $\tau_{K,N}^{(t)}(\cdot)$ is not non-decreasing. Now that we know the bound $\sqrt{\frac{N-1}{K}}\pi + h$ for the diameter of M, obviously $(\mathsf{d}(x_0, x_1) - h)_+$ is in $\left[0, \sqrt{\frac{N-1}{K}}\pi\right]$, where $\tau_{K,N}^{(t)}(\cdot)$ is non-decreasing. $\square$

2.4 Stability under convergence

Like in the case of the rough curvature bound we can prove a stability result that shows we can pass from discrete spaces to continuous limit spaces.

Theorem 2.4.1. *Let (M, d, m) be a normalized metric measure space and consider $\{(M_h, \mathsf{d}_h, m_h)\}_{h>0}$ a family of normalized metric measure spaces such that for each $h > 0$ the space (M_h, d_h, m_h) satisfies the rough curvature-dimension condition h-$\mathsf{CD}(K_h, N_h)$ and has diameter L_h for some real numbers K_h, N_h and L_h with $N_h \geq 1$ and $L_h > 0$. Assume that for $h \to 0$ we have*

$$(M_h, \mathsf{d}_h, m_h) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$$

and $(K_h, N_h, L_h) \to (K, N, L)$ for some $(K, N, L) \in \mathbb{R}^3$ satisfying $K \cdot L^2 < (N-1)\pi^2$. Then the space (M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(K, N)$ in the sense of the Definition 2.1.3 and has diameter $\leq L$.

For given numbers $h \geq 0$, $t \in [0, 1]$, $K \in \mathbb{R}$ and $N \geq 1$ we use the notations

$$\begin{aligned} T_{h,K,N}^{(t)}(q|m) \;:=\; -\int \Big[&\tau_{K,N'}^{(1-t)}((\mathsf{d}(x_0, x_1) - \delta_K h)_+) \cdot \rho_0^{-1/N'}(x_0) \\ &+ \tau_{K,N'}^{(t)}((\mathsf{d}(x_0, x_1) - \delta_K h)_+) \cdot \rho_1^{-1/N'}(x_1) \Big] \, dq(x_0, x_1) \end{aligned}$$

and

$$T_{K,N}^{(t)}(q|m) := T_{0,K,N}^{(t)}(q|m),$$

whenever q is a δh-coupling of $\nu_0 = \rho_0 \cdot m$ and $\nu_1 = \rho_1 \cdot m$. Recall that $\delta = 1$ for $K \geq 0$ and $\delta = -1$ for $K < 0$.

Lemma 3.3 from [St06b] shows that $T_{K,N}^{(t)}(\cdot|m)$ is upper semicontinuous. The next result gives the upper semicontinuity of $T_{h,K,N}^{(t)}(\cdot|m)$ for arbitrary $h \geq 0$.

Lemma 2.4.2. *Let $h > 0$, $t \in [0,1]$, $K \in \mathbb{R}$ and $N \geq 1$ be given. For any sequence $\{q^{(k)}\}_{k\in\mathbb{N}}$ of couplings with the same marginals ν_0 and ν_1, converging weakly to some coupling $q^{(\infty)}$, we have*

$$\limsup_{k\to\infty} T^{(t)}_{h,K,N}(q^{(k)}|m) \leq T^{(t)}_{h,K,N}(q^{(\infty)}|m) \tag{2.4.1}$$

Proof. Consider $\{q^{(k)}\}_{k\in\mathbb{N}}$ and $q^{(\infty)}$ like in the statement. It is sufficient to prove that

$$\begin{aligned}\liminf_{k\to\infty} \quad & \int \tau^{(1-t)}_{K,N'}((\mathsf{d}(x_0,x_1)-\delta_K h)_+)\cdot \rho_0^{-1/N'}(x_0)dq^{(k)}(x_0,x_1)\\ & \geq \int \tau^{(1-t)}_{K,N'}((\mathsf{d}(x_0,x_1)-\delta_K h)_+)\cdot \rho_0^{-1/N'}(x_0)dq^{(\infty)}(x_0,x_1),\end{aligned} \tag{2.4.2}$$

because then a similar inequality will take place with ρ_1 instead of ρ_0 and t instead of $1-t$ and by summing up the two inequalities we will get (2.4.1).

For $k \in \mathbb{N}\cup\{\infty\}$ denote by $Q^{(k)}(x_0,dx_1)$ the disintegration of $dq^{(k)}(x_0,x_1)$ with respect to $d\nu_0(x_0)$. If $C \in \mathbb{R}_+\cup\{\infty\}$ put

$$\vartheta^{(k)}_C(x_0) = \int \left[\tau^{(1-t)}_{K,N'}((\mathsf{d}(x_0,x_1)-\delta_K h)_+)\wedge C\right] Q^{(k)}(x_0,dx_1).$$

Consider now $C \in \mathbb{R}_+$ fixed. The space $\mathcal{C}_b(M)$ of continuous and bounded functions is dense in $L_1(M,\nu_0)$ and therefore for each $\epsilon > 0$ one can find a function $\varphi \in \mathcal{C}_b(M)$ such that

$$\int C\cdot\left|\left[\rho_0^{-1/N}\wedge C\right]-\varphi\right| d\nu_0 \leq \epsilon.$$

This together with the fact that $0 \leq \vartheta^{(k)}_C \leq C$ implies that for all $k \in \mathbb{N}\cup\{\infty\}$ we have

$$\int \vartheta^{(k)}_C\cdot\left|\left[\rho_0^{-1/N}\wedge C\right]-\varphi\right| d\nu_0 \leq \epsilon. \tag{2.4.3}$$

The sequence $\{q^{(k)}\}_{k\in\mathbb{N}}$ converges weakly to $q^{(\infty)}$ on $M\times M$ and since the function $(x_0,x_1)\mapsto \tau^{(1-t)}_{K,N'}((\mathsf{d}(x_0,x_1)-\delta_K h)_+)\wedge C$ lies in $\mathcal{C}_b(M\times M)$ there exists a $k(\epsilon)\in\mathbb{N}$ such that for each $k \geq k(\epsilon)$

$$\int \vartheta^{(\infty)}_C \varphi\, d\nu_0 \leq \int \vartheta^{(k)}_C \varphi\, d\nu_0 + \epsilon. \tag{2.4.4}$$

Thus, for each $k \geq k(\epsilon)$ we obtain

$$\begin{aligned}\int \vartheta^{(\infty)}_C\cdot\left[\rho_0^{-1/N}\wedge C\right] d\nu_0 \quad & \leq \quad \int \vartheta^{(\infty)}_C\cdot\left|\left[\rho_0^{-1/N}\wedge C\right]-\varphi\right| d\nu_0 + \int \vartheta^{(\infty)}_C\cdot\varphi\, d\nu_0\\ & \overset{(2.4.3)}{\leq} \quad \int \vartheta^{(\infty)}_C\cdot\varphi\, d\nu_0 + \epsilon \overset{(2.4.4)}{\leq} \int \vartheta^{(k)}_C\cdot\varphi\, d\nu_0 + 2\epsilon\\ & \overset{(2.4.3)}{\leq} \quad \int \vartheta^{(k)}_C\cdot\left[\rho_0^{-1/N}\wedge C\right] d\nu_0 + 3\epsilon\\ & \leq \quad \int \vartheta^{(k)}_\infty\cdot\rho_0^{-1/N} d\nu_0 + 3\epsilon.\end{aligned}$$

This leads to

$$\int \vartheta_C^{(\infty)} \cdot \left[\rho_0^{-1/N} \wedge C\right] d\nu_0 \leq \liminf_{k\to\infty} \int \vartheta_\infty^{(k)} \cdot \rho_0^{-1/N} d\nu_0$$

for each $C \in \mathbb{R}_+$. Now if we let C tend to ∞, by monotone convergence we obtain

$$\int \vartheta_\infty^{(\infty)} \cdot \rho_0^{-1/N} d\nu_0 \leq \liminf_{k\to\infty} \int \vartheta_\infty^{(k)} \cdot \rho_0^{-1/N} d\nu_0,$$

which gives (2.4.2). □

Proof of Theorem 2.4.1. Let $\{(M_h, \mathsf{d}_h, m_h)\}_{h>0}$ be a family of normalized metric measure spaces, each (M_h, d_h, m_h) satisfying a rough curvature-dimension condition h-$\mathsf{CD}(K_h, N_h)$ and having diameter $\leq L_h$. Suppose that $\{(M_h, \mathsf{d}_h, m_h)\}_{h>0}$ converges to some metric measure space (M, d, m) in the metric $\mathbb{D}$ as $h \to 0$. Then the limit space (M, d, m) must have diameter $\leq L$. Without loss of generality, one can assume that $N_h > 1$ and that there exists a triple (K_0, N_0, L_0) with $K_h \leq K_0$, $N_h \geq N_0$, $L_h \leq L_0$ for all $h > 0$ and with $K_0 \cdot L_0^2 < (N_0 - 1)\pi^2$.

In order to prove the curvature-dimension condition $\mathsf{CD}(K, N)$ let $\nu_0, \nu_1 \in \mathcal{P}_2(M, \mathsf{d}, m)$ be an arbitrary pair of measures with $\nu_i = \rho_i \cdot m$, $i = 0, 1$. Let a number $\epsilon > 0$ be given. We fix an arbitrary optimal coupling $\hat{q}$ of ν_0 and ν_1 and for $r \in \mathbb{R}_+$ denote

$$\begin{aligned} D_r &:= \{(x_0, x_1) \in M \times M : \rho_0(x_0) < r, \rho_1(x_1) < r\} \\ \alpha_r &:= \hat{q}(D_r) \\ \hat{q}^{(r)}(\cdot) &:= \frac{1}{\alpha_r} \hat{q}^{(r)}(\cdot \cap D_r). \end{aligned}$$

The measure $\hat{q}^{(r)}$ has marginals

$$\hat{\nu}_0^{(r)}(\cdot) := \hat{q}^{(r)}(\cdot \times M), \qquad \hat{\nu}_1^{(r)}(\cdot) := \hat{q}^{(r)}(M \times \cdot)$$

with bounded densities. For sufficiently large $r = r(\epsilon)$ we have also

$$\mathsf{d}_W(\nu_0, \hat{\nu}_0^{(r)}) \leq \epsilon, \qquad \mathsf{d}_W(\nu_1, \hat{\nu}_1^{(r)}) \leq \epsilon. \tag{2.4.5}$$

Since the space (M, d, m) has finite diameter and the densities of $\hat{\nu}_0^{(r)}$ and $\hat{\nu}_1^{(r)}$ are bounded, one can find a number $R \in \mathbb{R}$ such that

$$\sup_{i=0,1} \operatorname{Ent}(\hat{\nu}_i^{(r)}|m) + \frac{\sup_{h>0}|K_h|}{8} \left[\mathsf{d}_W(\hat{\nu}_0^{(r)}, \hat{\nu}_1^{(r)}) + 3\epsilon\right]^2 \leq R. \tag{2.4.6}$$

According to our hypothesis, $(M_h, \mathsf{d}_h, m_h) \xrightarrow{\mathbb{D}} (M, \mathsf{d}, m)$ as $h \to 0$, therefore one can choose $h = h(\epsilon) \in (0, \epsilon)$ and a coupling $\hat{\mathsf{d}}$ of the metrics d and d_h such that

$$\frac{1}{2}\hat{\mathsf{d}}_W(m_h, m) \leq \mathbb{D}((M_h, \mathsf{d}_h, m_h), (M, \mathsf{d}, m)) \leq \left(\frac{\epsilon}{4C}\right) \wedge \exp\left(-\frac{2 + 4L_0^2 R}{\epsilon^2}\right), \tag{2.4.7}$$

where the constant C is to be specified later. Fix now a coupling p of m and m_h which is optimal with respect to $\hat{\mathsf{d}}$ and consider P and P' the disintegrations of p with respect to m and m_h respectively. Like in Lemma 4.19 in [St06a], P' induces a canonical map $P' : \mathcal{P}_2(M, \mathsf{d}, m) \to \mathcal{P}_2(M_h, \mathsf{d}_h, m_h)$. Put

$$\nu_{i,h} := P'(\hat{\nu}_i^{(r)}) = \rho_{i,h} \cdot m_h$$

with

$$\rho_{i,h}(y) = \int_M \hat{\rho}_i^{(r)}(x) P'(y, dx) \text{ for } i = 0, 1.$$

By applying Lemma 4.19 from [St06a] we obtain in turn

$$\hat{\mathsf{d}}_W(\hat{\nu}_i^{(r)}, \nu_{i,h})^2 \overset{(2.4.5)}{\leq} \frac{2 + 4L_0^2 R}{-\log \mathbb{D}((M_h, \mathsf{d}_h, m_h), (M, \mathsf{d}, m))} \overset{(2.4.6)}{\leq} \epsilon^2 \tag{2.4.8}$$

and

$$\mathrm{Ent}(\nu_{i,h}|m_h) \leq \mathrm{Ent}(\hat{\nu}_i^{(r)}|m) \tag{2.4.9}$$

for $i = 0, 1$.

The approximating space (M_h, d_h, m_h) satisfies the rough curvature-dimension condition h-$\mathsf{CD}(K_h, N_h)$, which ensures the existence of a $\delta_h h$-optimal coupling q_h of $\nu_{0,h}$ and $\nu_{1,h}$ and for each $t \in [0, 1]$ the existence of an h-rough t-intermediate point $\eta_{t,h} \in \mathcal{P}_2(M_h, \mathsf{d}_h, m_h)$ of $\nu_{0,h}$ and $\nu_{1,h}$ satisfying

$$S_{N'}(\eta_{t,h}|m_h) \leq T^{(t)}_{h,K',N'}(q_h|m_h) \tag{2.4.10}$$

for all $K' \leq K_h$ and $N' \geq N_h$. Lemma 4.19 from [St06a] gives also a canonical map $P : \mathcal{P}_2(M_h, \mathsf{d}_h, m_h) \to \mathcal{P}_2(M, \mathsf{d}, m)$. Put now

$$\Gamma_t^\epsilon := P(\eta_{t,h}) \tag{2.4.11}$$

with $h = h(\epsilon)$ as above. Recall that P is defined such that the density of Γ_t^ϵ with respect to m is given by

$$\rho_t^\epsilon(x) = \int_{M_h} \rho_{t,h}(y) P(dy, x),$$

with $\rho_{t,h}$ being the density of $\eta_{t,h}$ with respect to m_h. Applying now Jensen's inequality to the convex function $r \mapsto -r^{1-1/N}$ we have

$$\begin{aligned} S_{N'}(\Gamma_t^\epsilon|m) &= -\int_M (\rho_t^\epsilon)^{1-1/N'} dm = -\int_M \left[\int_{M_h} \rho_{t,h}(y) P(dy, x)\right]^{1-1/N'} dm(x) \\ &\leq -\int_M \int_{M_h} \rho_{t,h}(y)^{1-1/N'} P(dy, x) dm(x) = \int_{M_h} \rho_{t,h}(y)^{1-1/N'} dm_h(y) \\ &= S_{N'}(\eta_{t,h}|m_h), \end{aligned}$$

so we have obtained

$$S_{N'}(\Gamma_t^\epsilon|m) \leq S_{N'}(\eta_{t,h}|m_h) \tag{2.4.12}$$

for all $N' \geq N_h$ and all $t \in [0,1]$. Proposition 2.2.7 (iv) shows that the rough curvature-dimension condition h-$\mathsf{CD}(K_h, N_h)$ for the space $(M_h,\, \mathsf{d}_h, m_h)$ implies the rough curvature bound h-$\mathbb{C}\mathrm{urv}(M_h,\, \mathsf{d}_h, m_h) \geq K_h$. This entails

$$\begin{aligned}
\mathrm{Ent}(\Gamma_t^\epsilon|m) \quad &\leq \quad \mathrm{Ent}(\eta_{t,h}|m_h) \\
&\leq \quad (1-t)\mathrm{Ent}(\nu_{0,h}|m_h) + t\mathrm{Ent}(\nu_{1,h}|m_h) \\
&\qquad\qquad - \frac{K_h}{2} t(1-t)\, \hat{\mathsf{d}}_W^{\delta_h h}(\nu_{0,h}, \nu_{1,h})^2 \\
&\overset{\text{Lemma 1.2.5}}{\leq} \quad \sup_{i=0,1} \mathrm{Ent}(\nu_{i,h}|m_h) + \frac{\sup_{h>0}|K_h|}{8}\Big[\hat{\mathsf{d}}_W(\nu_{0,h}, \nu_{1,h}) + h\Big]^2 \\
&\overset{(2.4.8),(2.4.9)}{\leq} \quad \sup_{i=0,1} \mathrm{Ent}(\hat{\nu}_i^{(r)}|m) + \frac{\sup_{h>0}|K_h|}{8}\Big[\hat{\mathsf{d}}_W(\hat{\nu}_0^{(r)}, \hat{\nu}_1^{(r)}) + 2\epsilon + h\Big]^2 \\
&\leq \quad \sup_{i=0,1} \mathrm{Ent}(\hat{\nu}_i^{(r)}|m) + \frac{\sup_{h>0}|K_h|}{8}\Big[\hat{\mathsf{d}}_W(\hat{\nu}_0^{(r)}, \hat{\nu}_1^{(r)}) + 3\epsilon\Big]^2 \\
&\overset{(2.4.6)}{\leq} \quad R.
\end{aligned}$$

Together with (2.4.7), this implies again by Lemma 4.19 from [St06a] that

$$\hat{\mathsf{d}}_W(\Gamma_t^\epsilon, \eta_{t,h}) \leq \epsilon. \tag{2.4.13}$$

Let Q_h and Q'_h be the disintegrations of q_h with respect to $\nu_{0,h}$ and $\nu_{1,h}$ respectively. For $h = h(\epsilon)$ as above and for fixed K', N' and $t \in [0,1]$ put

$$v_0(y_0) := \int_{M_h} \tau_{K',N'}^{(1-t)}((\,\mathsf{d}_h(y_0,y_1) - \delta_{K'}h)_+)\; Q_h(y_0, dy_1)$$

and

$$v_1(y_1) := \int_{M_h} \tau_{K',N'}^{(t)}((\,\mathsf{d}_h(y_0,y_1) - \delta_{K'}h)_+)\; Q'_h(dy_0, y_1).$$

Then from Jensen's inequality we have

$$\begin{aligned}
-T_{h,K',N'}^{(t)}(q_h|m_h) \;&=\; \sum_{i=0}^1 \int_{M_h} \rho_{i,h}(y)^{1-1/N'} \cdot v_i(y)\; dm_h(y) \\
&=\; \sum_{i=0}^1 \int_{M_h} \Big[\int_M \hat{\rho}_i^{(r)}(x)\; P'(y,dx)\Big]^{1-1/N'} \cdot v_i(y)\; dm_h(y) \\
&\geq\; \sum_{i=0}^1 \int_{M_h} \int_M \Big[\hat{\rho}_i^{(r)}(x)\Big]^{1-1/N'} \cdot v_i(y)\; P'(y,dx)\; dm_h(y) \\
&=\; \sum_{i=0}^1 \int_{M_h} \Big[\hat{\rho}_i^{(r)}(x)\Big]^{1-1/N'} \Big[\int_{M_h} v_i(y)\; P(x,dy)\Big]\, dm(y).
\end{aligned}$$

Now

$$\begin{aligned}\int_{M_h} v_0(y_0)P(x_0,dy_0) &= \int_{M_h}\int_{M_h} \tau^{(1-t)}_{K',N'}((\mathsf{d}_h(y_0,y_1)-\delta_{K'}h)_+)Q_h(y_0,dy_1)P(x_0,dy_0)\\
&\geq \int_{M_h}\int_{M_h}\int_M \left[\tau^{(1-t)}_{K',N'}(\mathsf{d}(x_0,x_1)) - C\cdot\Big((\mathsf{d}_h(y_0,y_1)-\delta_{K'}h)_+ - \mathsf{d}(x_0,x_1)\Big)\right]\\
&\qquad\cdot\frac{\hat\rho_1^{(r)}(x_1)}{\rho_{1,h}(y_1)}P'(y_1,dx_1)\ Q_h(y_0,dy_1)\ P(x_0,dy_0)\\
&\geq \int_{M_h}\int_{M_h}\int_M \left[\tau^{(1-t)}_{K',N'}(\mathsf{d}(x_0,x_1)) - C\cdot\Big(\mathsf{d}_h(y_0,y_1) - \mathsf{d}(x_0,x_1)+h\Big)\right]\\
&\qquad\cdot\frac{\hat\rho_1^{(r)}(x_1)}{\rho_{1,h}(y_1)}P'(y_1,dx_1)\ Q_h(y_0,dy_1)\ P(x_0,dy_0)\\
&\geq \int_{M_h}\int_{M_h}\int_M \left[\tau^{(1-t)}_{K',N'}(\mathsf{d}(x_0,x_1)) - C\cdot\Big(\hat{\mathsf{d}}(x_0,y_0) + \hat{\mathsf{d}}(x_1,y_1)+h\Big)\right]\\
&\qquad\cdot\frac{\hat\rho_1^{(r)}(x_1)}{\rho_{1,h}(y_1)}P'(y_1,dx_1)\ Q_h(y_0,dy_1)\ P(x_0,dy_0)\end{aligned}$$

where

$$C := \max\left\{\frac{\partial}{\partial\theta}\tau^{(s)}_{K',N'}(\theta) : s\in[0,1],\ K'\leq K_0,\ N'\geq N_0,\ \theta\leq L_0\right\}.$$

In a similar way, we have the estimate

$$\begin{aligned}&\int_{M_h} v_1(y_1)P(x_1,dy_1)\\
&\qquad\geq \int_{M_h}\int_{M_h}\int_M \left[\tau^{(t)}_{K',N'}(\mathsf{d}(x_0,x_1)) - C\Big(\hat{\mathsf{d}}(x_0,y_0) + \hat{\mathsf{d}}(x_1,y_1)+h\Big)\right]\\
&\qquad\qquad\cdot\frac{\hat\rho_0^{(r)}(x_0)}{\rho_{0,h}(y_0)}P'(y_0,dx_0)\ Q'_h(y_1,dy_0)\ P(x_1,dy_1).\end{aligned}$$

Consider the measure

$$\begin{aligned}d\overline{q}^{(r)}(x_0,x_1) &:= \int_{M_h\times M_h}\frac{\hat\rho_0^{(r)}(x_0)\hat\rho_1^{(r)}(x_1)}{\rho_{0,h}(y_0)\rho_{1,h}(y_1)}P'(y_1,dx_1)P'(y_0,dx_0)dq_h(y_o,y_1)\\
&= \int_{M_h\times M_h}\frac{\hat\rho_0^{(r)}(x_0)\hat\rho_1^{(r)}(x_1)}{\rho_{1,h}(y_1)}P'(y_1,dx_1)Q_h(y_0,dy_1)P(x_0,dy_0)m(dx_0).\end{aligned}$$

Then $\overline{q}^{(r)}$ is a (not necessarily optimal) coupling of $\hat{q}_0^{(r)}$ and $\hat{q}_0^{(r)}$. Consider also a coupling q^ϵ of ν_0 and ν_1 given by

$$q^\epsilon(A) := \alpha_r \overline{q}^{(r)} + \hat{q}(A \cap (M \times M \setminus E_r))$$

for any $A \subset M \times M$ measurable and for $r = r(\epsilon)$. From the above estimates we obtain

$$\begin{aligned} T^{(t)}_{h,K',N'}(q_h|m_h) &\leq T^{(t)}_{K',N'}(\overline{q}^{(r)}|m) \\ &\qquad + C\int_M \left[\hat{\rho}_0^{(r)}(x)^{1-1/N'} + \hat{\rho}_1^{(r)}(x)^{1-1/N'}\right](\hat{\mathsf{d}}(x,y)+h)dp(x,y) \\ &\leq T^{(t)}_{K',N'}(\overline{q}^{(r)}|m) + 2C\,\hat{\mathsf{d}}_W(m,m_h) + h \leq T^{(t)}_{K',N'}(\overline{q}^{(r)}|m) + 2\epsilon, \end{aligned}$$

by using (2.4.7). We also have

$$\lim_{\epsilon\to 0}\left|T^{(t)}_{K',N'}(q^\epsilon|m) - T^{(t)}_{K',N'}(\overline{q}^{(r(\epsilon))}|m)\right| = 0. \tag{2.4.14}$$

In this way, for each $\epsilon > 0$ we have found a probability measure q^ϵ on $M \times M$ and a family of probability measures $\{\Gamma_t^\epsilon\}_{t\in[0,1]}$ on M such that

$$S_{N'}(\Gamma_t^\epsilon|m) \overset{(2.4.12)}{\leq} S_{N'}(\eta_{t,h}|m_h) \overset{(2.4.10)}{\leq} T^{(t)}_{h,K',N'}(q_h|m_h) \overset{(2.4.14)}{\leq} T^{(t)}_{K',N'}(\overline{q}^{(r(\epsilon))}|m) + 2\epsilon. \tag{2.4.15}$$

The fact that M is compact implies that there exists a sequence $(\epsilon(k))_{k\in\mathbb{N}}$ converging to 0 such that the measures $q^{\epsilon(k)}$ tend to some measure q and for each $t \in [0,1]\cap\mathbb{Q}$ the probability measures $\Gamma_t^{\epsilon(k)}$ converge to some Γ_t. Since all q^ϵ are couplings of ν_0 and ν_1, the measure q is also a coupling of ν_0 and ν_1. Moreover, (2.4.5), (2.4.8) and (2.4.13) yield that q is in fact an optimal coupling.

For each $h > 0$ and $t \in [0,1]$ the measure $\eta_{t,h}$ is an h-rough t-intermediate point between $\nu_{0,h}$ and $\nu_{1,h}$ in $\mathcal{P}_2(M_h, \mathsf{d}_h, m_h)$. But $\nu_{0,h}$ and $\nu_{1,h}$ converge to ν_0 and ν_1 respectively, as $h \to 0$. Together with (2.4.13), this leads to

$$\begin{aligned} \mathsf{d}_W(\nu_0,\Gamma_t) &\leq t\,\mathsf{d}_W(\nu_0,\nu_1) \\ \mathsf{d}_W(\Gamma_t,\nu_1) &\leq (1-t)\,\mathsf{d}_W(\nu_0,\nu_1) \end{aligned}$$

for any $t \in [0,1]\cap\mathbb{Q}$. Therefore, the family $\{\Gamma_t\}_t$ extends to a geodesic in $\mathcal{P}_2(M, \mathsf{d}, m)$ connecting ν_0 and ν_1. Since $S_{N'}(\cdot|m)$ is lower semicontinuous (Lemma 2.1.1) and $T^{(t)}_{K',N'}(\cdot|m)$ is upper semicontinuous, the estimate (2.4.14) implies

$$S_{N'}(\Gamma_t|m) \leq \liminf_{k\to\infty} S_{N'}(\Gamma_t^{\epsilon(k)}|m) \leq \liminf_{k\to\infty} T^{(t)}_{K',N'}(q^{\epsilon(k)}|m) \leq T^{(t)}_{K',N'}(q|m)$$

for all $t \in [0,1]$, all $N' > N = \lim_{h\to 0} N_h$ and all $K' > K = \lim_{h\to 0} K_h$. The inequality $S_{N'}(\Gamma_t|m) \leq T^{(t)}_{K',N'}(q|m)$ holds also for $K' = K$ and $N' = N$, by the continuity of $S_{N'}$ and $T^{(t)}_{K',N'}$ in (K', N'). This ends the proof of the theorem.

□

2.5 Stability under discretization

In this section we shall prove that the rough curvature-dimension condition is preserved if we consider discretizations of a geodesic metric measure space fulfilling a curvature-dimension condition in the sense of the Definition 2.1.3.

Theorem 2.5.1. *Let (M, d, m) be a metric measure space that satisfies the curvature-dimension condition $\mathsf{CD}(K, N)$ for some real numbers K and $N \geq 1$. Then for each $h > 0$ any discretization (M_h, d, m_h) with $R(h) \leq h/4$ satisfies the rough curvature-dimension condition h-$\mathsf{CD}(K, N)$.*

Proof. Assume that (M, d, m) satisfies the curvature-dimension condition $\mathsf{CD}(K, N)$ and consider a discretization (M_h, d, m_h) with $M_h = \{x_j : j \geq 1\} \subset M$. Suppose that $\{A_j\}_{j\geq 1}$ is the corresponding covering of M with mutually disjoint sets such that $x_j \in A_j$, $m_h(\{x_j\}) = m(A_j)$ and $\operatorname{diam}(A_j) \leq R(h)$ for each $j \geq 1$. In order to check the rough curvature-dimension condition h-$\mathsf{CD}(K, N)$ for the discrete space (M_h, d, m_h), let ν_1^h, ν_2^h be a pair of arbitrarily given measures in $\mathcal{P}_2(M_h, \mathsf{d}, m_h)$, say

$$\nu_i^h = \left(\sum_{j=1}^{\infty} a_{i,j}^h 1_{\{x_j\}}\right) \cdot m_h, \qquad i = 1, 2.$$

Put

$$\nu_i := \left(\sum_{j=1}^{\infty} a_{i,j}^h 1_{A_j}\right) \cdot m, \qquad i = 1, 2.$$

Applying the $\mathsf{CD}(K, N)$ property assumed to be true for the space (M, d, m), one can obtain an optimal coupling q of ν_1 and ν_2 and for each $t \in [0, 1]$ a t-intermediate point η_t of ν_1 and ν_2 such that (2.1.1) takes place for any $N' \geq N$. Assume that $\eta_t = \rho_t m$.

The formula

$$\eta_t^h(\{x_j\}) := \eta_t(A_j), \qquad j = 1, 2, \ldots$$

defines for each $t \in [0, 1]$ a probability measure on M_h which is absolutely continuous with respect to m_h with density

$$\rho_t^h = \sum_{j=1}^{\infty} \frac{\eta_t(A_j)}{m(A_j)} 1_{A_j} = \sum_{j=1}^{\infty} \frac{\int_{A_j} \rho_t \, dm}{m(A_j)} 1_{A_j}.$$

Hence for $N' \geq N$ we have, due to Jensen's inequality,

$$\begin{aligned} S_{N'}(\eta_t^h|m_h) &= -\sum_{j=1}^{\infty}\left(\frac{1}{m(A_j)}\int_{A_j}\rho_t\ dm\right)^{-1/N'} m_h(\{x_j\}) \\ &\leq -\sum_{j=1}^{\infty}\frac{1}{m(A_j)}\left(\int_{A_j}\rho_t^{-1/N'}\ dm\right) m_h(\{x_j\}) \\ &= -\sum_{j=1}^{\infty}\int_{A_j}\rho_t^{-1/N'}\ dm = S_{N'}(\eta_t|m). \end{aligned}$$

Therefore we get

$$\begin{aligned} S_{N'}(\eta_t^h|m_h) \ \leq\ -\int \Big[&\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1))\cdot\rho_0^{-1/N'}(x_0) \\ &+\tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1))\cdot\rho_1^{-1/N'}(x_1)\Big]\, dq(x_0,x_1). \end{aligned} \tag{2.5.1}$$

Suppose that we are in the case $K < 0$. Let q^h be a $-2R(h)$-optimal coupling of ν_0^h and ν_1^h. Denote

$$\widehat{q} := \sum_{j,k=1}^{n}\left[q^h(\{(x_j,x_k)\})\delta_{(x_j,x_k)} \times \frac{1_{A_j\times A_k}}{m(A_j)m(A_k)}(m\times m)\right].$$

Then $\widehat{q}$ is a measure on $M_h \times M_h \times M \times M$ which has marginals ν_0^h, ν_1^h, ν_0 and ν_1. Moreover, the projection of $\widehat{q}$ on the first two factors is equal to q^h.

For $K < 0$, $N > 1$ and arbitrarily fixed $t \in (0,1)$ the function $\tau_{K,N}^{(1-t)}(\cdot)$ is non-increasing on $[0,\infty)$, therefore we have

$$\begin{aligned} &-\int_{M\times M}\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1))\cdot\rho_0^{-1/N'}(x_0)dq(x_0,x_1) \\ &= -\int_{M_h\times M_h\times M\times M}\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1))\cdot\rho_0^{-1/N'}(x_0)d\widehat{q}(x_0^h,x_1^h,x_0,x_1) \\ &\leq -\int_{M_h\times M_h\times M\times M}\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_0^h)+\mathsf{d}(x_0^h,x_1^h)+\mathsf{d}(x_1^h,x_1)) \\ &\qquad\qquad\cdot\rho_0^{-1/N'}(x_0)d\widehat{q}(x_0^h,x_1^h,x_0,x_1) \\ &= \sum_{j,k}\frac{q^h(\{(x_j,x_k)\})}{m(A_j)m(A_k)}\int_{A_j\times A_k}\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,x_j)+\mathsf{d}(x_j,x_k)+\mathsf{d}(x_k,y)) \\ &\qquad\qquad\cdot\left(a_{0,j}^h\right)^{-1/N'}dm(x)dm(y) \\ &\leq \sum_{j,k}\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_j,x_k)+2R(h))\cdot\left(a_{0,j}^h\right)^{-1/N'}q^h(\{(x_j,x_k)\}). \end{aligned}$$

In a similar way we can majorize the second term of the integral in (2.5.1) and obtain the desired inequality for the $-2R(h)$-optimal coupling $\widehat{q}$ of ν_0^h and ν_1^h and for η_t^h.

If $K = 0$ then it is easy to see that $S_{N'}(\nu_i^h|m_h) = S_{N'}(\nu_i|m)$, $i = 1, 2$, which gives directly

$$S_{N'}(\eta_t^h|m_h) \leq (1-t) \cdot S_{N'}(\nu_0^h|m_h) + t \cdot S_{N'}(\nu_1^h|m_h)$$

for all $N' \geq N$.

For $K > 0$, $N > 1$ and arbitrarily fixed $t \in (0, 1)$, since Theorem 2.3.5 gives us a bound for the diameter for which $\tau_{K,N}^{(t)}(\cdot)$ is actually non-decreasing, so that $\tau_{K,N}^{(t)}((\mathsf{d}(x_0, x_1) - \alpha)_+)$ is non-increasing in α and then the proof goes like in the case $K < 0$.

Like in the proof of Theorem 1.3.1 one can show that η_t^h is at least a $4R(h)$-rough t-intermediate point of ν_1^h, ν_2^h. Therefore, if $h \geq 4R(h)$ the discretization (M_h, d, m_h) satisfies the rough curvature-dimension condition h-$CD(K, N)$.

$\square$

The above result provides a series of examples that we are already familiar with, from the previous chapter.

(i) The space $\mathbb{Z}^n$ with the metric d_1, which is the restriction of the norm $|\cdot|_1$ in $\mathbb{R}^n$, and with the measure $\overline{m}_n = \sum_{x \in \mathbb{Z}^n} \delta_x$ satisfies h-$\mathsf{CD}(0, n)$ for any $h \geq 2n$.

(ii) The n-dimensional grid $\mathbb{E}^n$ having $\mathbb{Z}^n$ as set of vertices, equipped with the graph distance and with the measure m_n which is the 1-dimensional Lebesgue measure on the edges, satisfies h-$\mathsf{CD}(0, n)$ for any $h \geq 2(n+1)$.

(iii) Let G be the graph that tiles the euclidian plane with equilateral triangles of edge r, with the graph metric d_G and with the 1-dimensional Lebesgue measure m on the edges. Then G fulfills the h-$\mathsf{CD}(0, 2)$ condition for any $h \geq 8r\sqrt{3}/3$.

(iv) The graph G' that tiles the euclidian plane with regular hexagons of edge length r, equipped with the graph metric $\mathsf{d}_{G'}$ and with the 1-dimensional measure m', satisfies h-$\mathsf{CD}(0, 2)$ for any $h \geq 34r/3$.

(v) (The homogeneous planar graphs). For any numbers $l, n \geq 3$ with $\frac{1}{l} + \frac{1}{n} \leq \frac{1}{2}$ and for any $r > 0$ both metric measure spaces $(\mathbb{V}(l, n, r), \mathsf{d}, \widetilde{m})$ and $(\mathbb{G}(l, n, r), \mathsf{d}, m)$ defined in section 1.4 satisfy a rough curvature-dimension condition h-$\mathsf{CD}(K, 2)$ for $h \geq r \cdot C(l, n)$, where

$$K = \begin{cases} -\frac{1}{r^2}\left[\operatorname{arccosh}\left(2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1\right)\right]^2 & \text{for } \frac{1}{l} + \frac{1}{n} > \frac{1}{2} \\ \frac{1}{r^2}\left[\arccos\left(2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1\right)\right]^2 & \text{for } \frac{1}{l} + \frac{1}{n} < \frac{1}{2} \\ 0 & \text{for } \frac{1}{l} + \frac{1}{n} = \frac{1}{2} \end{cases}$$

and $C(l,n) = 4 \cdot \operatorname{arcsinh}\left(\frac{1}{sin\left(\frac{\pi}{n}\right)}\sqrt{\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1}\right) \Big/ \operatorname{arccosh}\left(2\frac{\cos^2\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi}{l}\right)} - 1\right)$.

Chapter 3

Dirichlet forms on graphs and their approximations

Within this chapter we turn our interest to some discrete/singular objects and their properties that might get lost partly from a large scale point of view and require a closer look on the singularities themselves.

We will be concerned with some classes of graph-like structures, whose transverse size is small but not zero, and we shall study approximations of ideal metric graphs with such graph-like objects in terms of heat kernels, Dirichlet forms, semigroups, resolvents and spectra. In their paper [KS03] K. Kuwae and T. Shioya studied convergence of operators and quadratic forms which are not necessarily defined on the same Hilbert space. They have developed a general theory of convergence of spectral structures, which defines the convergence of the whole machinery semigroups-resolvents-spectra-Dirichlet forms from the approximating sequence to the limit space. We will give a short overview of their theory in the first section.

In section 3.2 we consider various tubular open (bounded) domains that can approximate an edge, as basic building blocks for constructing further graph-like neighborhoods. Furthermore, we analyze in 3.3 the simplest example of metric graphs, the so-called spider, with a finite number of edges emanating from a vertex O. We consider approximations of such graphs with a certain class of graph-like open domains in $\mathbb{R}^3$ and prove convergence of the Laplace operators with Neumann boundary condition on the approximating domains towards the Laplacian on the graph with Kirchhoff boundary condition in the vertex O. Kuwae-Shioya theory will provide the convergence of the resolvents, semigroups, and spectra as well. This will permit us to extend our results to general finite graphs and some classes of graph-like neighborhoods. It is related to the recent work by P. Exner and O. Post [EP05], where the convergence of the spectra for some compact graph-like manifolds has been directly proved, by a completely different approach.

3.1 Preliminaries

The Gromov-Hausdorff convergence of nets of (isometry classes of) metric spaces was originally introduced in [Gro99] and further extended to compact metric measure spaces by K. Fukaya in [Fu87]. Since our spaces will not generally be compact, but only locally compact, we will work in the framework of *pointed* metric measure spaces, namely metric measure spaces with a distinguished basepoint.

Let $\mathcal{A}$ be a directed set. We denote by $\mathcal{M}$ the set of isomorphism classes of triples (M, p, m) with (M, p) a locally compact separable pointed metric space any bounded subset of which is relatively compact, and m a positive Radon measure on M.

Definition 3.1.1. *We say that the net $\{(M_\alpha, p_\alpha, m_\alpha)\}_{\alpha\in\mathcal{A}}$ of spaces in $\mathcal{M}$ converges in the sense of measured Gromov-Hausdorff convergence to a space $(M, p, m) \in \mathcal{M}$ if for every ball $B(p, r) \subset M$ there exists a sequence $\varepsilon_\alpha \searrow 0$ and there exist the balls $B(p_\alpha, r_\alpha) \subset M_\alpha$ with $r_\alpha \to r$ and the m_α-measurable maps $f_\alpha : B(p_\alpha, r_\alpha) \to B(p, r)$ (named ε_α-approximations) with the following properties:*

$$|d(f_\alpha(x), f_\alpha(y)) - d_\alpha(x, y)| < \varepsilon_\alpha, \forall x, y \in B(p_\alpha, r_\alpha), \forall \alpha \in \mathcal{A} \tag{3.1.1}$$

$$B(p, r) \subset B(f_\alpha(B(p_\alpha, r_\alpha)), \varepsilon_\alpha), \forall \alpha \in \mathcal{A} \tag{3.1.2}$$

$$\lim_\alpha \int_{B(p_\alpha r_\alpha)} u \circ f_\alpha \, dm_\alpha = \int_{B(p,r)} u \, dm, \forall u \in C_0(B(p, r)), \tag{3.1.3}$$

where d_α, d denote the distance functions on M_α, M respectively, $B(A, r) = \{y \in M : d(y, A) < r\}$ and where $C_0(A)$ is the set of real continuous functions on A with compact support in A.

Remark 3.1.2. The measured Gromov-Hausdorff convergence $(M_\alpha, p_\alpha, m_\alpha) \to (M, p, m)$ is equivalent to the existence of m_α-measurable ε_α-approximations $f_\alpha : B(p_\alpha, r_\alpha) \to B(p, r'_\alpha)$, $\varepsilon_\alpha \searrow 0$ such that $r_\alpha, r'_\alpha \nearrow \infty$, f_n satisfying (3.1.1) together with the following two properties

$$B(p, r'_\alpha) \subset B(f_\alpha(B(p_\alpha, r_\alpha)), \varepsilon_\alpha), \forall \alpha \in \mathcal{A} \tag{3.1.4}$$

$$\lim_\alpha \int_{B(p_\alpha r_\alpha)} u \circ f_\alpha \, dm_\alpha = \int_M u \, dm, \forall u \in C_0(M). \tag{3.1.5}$$

For $(M, p, m) \in \mathcal{M}$ we consider the space $L^2(M, m)$ with the usual inner product

$$(u, v)_{L_2} := \int_M uv \, dm, \qquad u, v \in L_2(M, m)$$

and define the norm $\|u\|_{L_2} := \sqrt{(u,u)_{L_2}}$ for any $u \in L_2(M,m)$.

Let us suppose in the sequel that $\{(M_\alpha, p_\alpha, m_\alpha)\}_{\alpha\in\mathcal{A}}$ is a net of spaces in $\mathcal{M}$ which converges in the sense of measured Gromov-Hausdorff convergence to a space $(M,p,m) \in \mathcal{M}$. Consider the m_α-measurable ε_α-approximations $f_\alpha : B(p_\alpha, r_\alpha) \to B(p, r'_\alpha)$ given by the previous remark, with $\varepsilon_\alpha \searrow 0$, $r_\alpha, r'_\alpha \nearrow \infty$. For $v \in C_0(\operatorname{supp} m)$ we define $\Phi_\alpha v := v \circ f_\alpha$ on $B(p_\alpha, r_\alpha)$ and $\Phi_\alpha v := 0$ on $M \setminus B(p_\alpha, r_\alpha)$.

Definition 3.1.3. *A net $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ with $u_\alpha \in L_2(M_\alpha, m_\alpha)$ (strongly) converges to an element $u \in L_2(M,m)$ if there exists a net $\{\widetilde{u}_\beta\}_{\beta\in\mathcal{B}}$ of functions in $C_0(\mathit{supp}\ m)$ tending to u in $L_2(M,m)$ such that*

$$\lim_\beta \limsup_\alpha \|\Phi_\alpha \widetilde{u}_\beta - u_\alpha\|_{L_2(M_\alpha,m_\alpha)} = 0 \tag{3.1.6}$$

Definition 3.1.4. *The net $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ with $u_\alpha \in L_2(M_\alpha,, m_\alpha)$ weakly converges to $u \in L_2(M,m)$ if*

$$\lim (u_\alpha, v_\alpha)_{L_2(M_\alpha,m_\alpha)} = (u,v)_{L_2(M,m)} \tag{3.1.7}$$

for any net $\{v_\alpha\}_{\alpha\in\mathcal{A}}$ with $v_\alpha \in L_2(M_\alpha, m_\alpha)$ tending strongly to a $v \in L_2(M,m)$.

The convergence of linear operators in L_2 with changing state spaces is defined as follows:

Definition 3.1.5. *(i) A net of linear operators $\{B_\alpha\}_{\alpha\in\mathcal{A}}$, each B_α on $L_2(M_\alpha, m_\alpha)$, $\alpha \in \mathcal{A}$, strongly converges to a linear operator B on $L_2(M,m)$ iff $B_\alpha u_\alpha \to Bu$ strongly for any net $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ with $u_\alpha \in L_2(M_\alpha, m_\alpha)$ strongly tending to $u \in L_2(M,m)$.*

(ii) We say that $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ compactly converges to B iff $B_\alpha u_\alpha \to Bu$ strongly for any net $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ with $u_\alpha \in L_2(M_\alpha, m_\alpha)$ weakly tending to $u \in L_2(M,m)$.

In the case of a fixed Hilbert space the concept of Γ-convergence was first introduced by De Giorgi. For more details we recommend the monograph Dal Maso [DM93]. Further, U. Mosco [Mos94] used a bilinear forms convergence method – related to Γ-convergence – in studying composite media problems; it is known in the literature as the Mosco convergence. The paper [KS03] extends both Γ-convergence and Mosco convergence to sequences of functions, respectively sequences of quadratic forms, defined on changing L_2-spaces.

Definition 3.1.6. (Γ-convergence). *We say that a net $\{F_\alpha : L_2(M_\alpha, m_\alpha) \to \overline{\mathbb{R}}\}_{\alpha\in\mathcal{A}}$ of functions Γ-converges to a function $F : L_2(M,m) \to \overline{\mathbb{R}}$ (or F is the Γ-limit of $\{F_\alpha\}_{\alpha\in\mathcal{A}}$) if and only if the following two properties hold:*

(1) For any net $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ with $u_\alpha \in L_2(M_\alpha, m_\alpha)$ which strongly converges to an element $u \in L_2(M,m)$ we have

$$F(u) \le \liminf_\alpha F_\alpha(u_\alpha); \tag{3.1.8}$$

(2) For any $u \in L_2(M,m)$ there exists a net $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ with $u_\alpha \in L_2(M_\alpha, m_\alpha)$ which strongly converges to u and

$$F(u) = \lim_{\alpha} F_\alpha(u_\alpha). \tag{3.1.9}$$

If $(M,p,m) \in \mathcal{M}$ we consider bilinear forms $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ which are positive definite and symmetric, $\mathcal{D}(\mathcal{E})$ being a (not necessarily dense) linear subspace of $L_2(M,m)$. We denote $\mathcal{E}_1(u,v) := (u,v)_{L_2} + \mathcal{E}(u,v)$, $u,v \in L_2(M,m)$, which is a (not necessarily complete) inner product on $\mathcal{D}(\mathcal{E})$. The bilinear form $\mathcal{E}$ is said to be closed if and only if $\mathcal{D}(\mathcal{E})$ is $\mathcal{E}_1$-complete. We shall identify a bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with the extended quadratic form $L_2(M,m) \ni u \mapsto \mathcal{E}(u,u) =: \mathcal{E}(u) \in \overline{\mathbb{R}}$ by setting $\mathcal{E}(u) := \infty$ for $u \in L_2(M,m) \setminus \mathcal{D}(\mathcal{E})$. Then, the bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closed if and only if the extended quadratic form $\mathcal{E} : L_2(M,m) \to \overline{\mathbb{R}}$ is lower semi-continuous.

According to Theorem 2.3 in [KS03] any net $\{\mathcal{E}_\alpha\}_{\alpha\in\mathcal{A}}$ of bilinear forms $\mathcal{E}_\alpha$ on $L_2(M_\alpha, m_\alpha)$ has a Γ-convergent subnet whose Γ-limit is a closed bilinear form.

Let $\{\mathcal{E}_\alpha\}_{\alpha\in\mathcal{A}}$ be a net of closed bilinear forms $\mathcal{E}_\alpha$ on $L_2(M_\alpha, m_\alpha)$ and $\mathcal{E}$ a closed bilinear form on $L_2(M,m)$.

Definition 3.1.7. (Mosco convergence). *We say that $\{\mathcal{E}_\alpha\}_{\alpha\in\mathcal{A}}$ Mosco converges to $\mathcal{E}$ if the following conditions are satisfied:*

(1) For any net $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ with $u_\alpha \in L_2(M_\alpha, m_\alpha)$ which weakly converges to an element $u \in L_2(M,m)$ we have

$$\mathcal{E}(u) \le \liminf_{\alpha} \mathcal{E}_\alpha(u_\alpha); \tag{3.1.10}$$

(2) For any $u \in L_2(M,m)$ there exists a net $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ with $u_\alpha \in L_2(M_\alpha, m_\alpha)$ which strongly converges to u and

$$\mathcal{E}(u) = \lim_{\alpha} \mathcal{E}_\alpha(u_\alpha). \tag{3.1.11}$$

Obviously, the Mosco convergence implies Γ-convergence.

Definition 3.1.8. (Asymptotic compactness). *The net $\{\mathcal{E}_\alpha\}_{\alpha\in\mathcal{A}}$ is called asymptotically compact if for any net $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ with $u_\alpha \in L_2(M_\alpha, m_\alpha)$ and such that $\limsup_\alpha(\mathcal{E}_\alpha(u_\alpha) + \|u_\alpha\|^2_{L_2(M_\alpha,m_\alpha)}) < \infty$, there exists a strongly convergent subnet $\{u_\alpha\}_{\alpha\in\mathcal{A}}$.*

According to Lemma 2.15 in [KS03], if $\{\mathcal{E}_\alpha\}_{\alpha\in\mathcal{A}}$ is asymptotically compact then $\{\mathcal{E}_\alpha\}_{\alpha\in\mathcal{A}}$ Mosco converges to $\mathcal{E}$ if and only if $\{\mathcal{E}_\alpha\}_{\alpha\in\mathcal{A}}$ Γ-converges to $\mathcal{E}$.

Definition 3.1.9. *We say that $\{\mathcal{E}_\alpha\}_{\alpha\in\mathcal{A}}$ compactly converges to $\mathcal{E}$ if $\{\mathcal{E}_\alpha\}_{\alpha\in\mathcal{A}}$ Mosco converges to $\mathcal{E}$ and $\{\mathcal{E}_\alpha\}_{\alpha\in\mathcal{A}}$ is asymptotically compact.*

Let L be the self-adjoint non-negative definite linear operator associated with the densely defined closed quadratic form $\mathcal{E}$, namely the infinitesimal generator associated with $\mathcal{E}$. Consider also the strongly continuous contraction semigroup $\{T_t\}_{t\geq 0}$ ($T_t = e^{-tL}$, $t \geq 0$) and the strongly continuous resolvent $\{R_\zeta\}_{\zeta\in\rho(L)}$ ($R_\zeta = (\zeta - L)^{-1}$, $\zeta \in \rho(L)$), where $\rho(L)$ is the resolvent set of L. If for each α we denote the similar corresponding objects by L_α, $\{T_t^\alpha\}_{t\geq 0}$, and $\{R_\zeta^\alpha\}_{\zeta\in\rho(L)}$ Theorem 2.4 and Corollary 2.5 in [KS03] state that

Theorem 3.1.10. *The following are equivalent:*

(i) $\mathcal{E}_\alpha \to \mathcal{E}$ *with respect to the Mosco topology (resp.* $\mathcal{E}_\alpha \to \mathcal{E}$ *compactly);*

(ii) $R_\zeta^\alpha \to R_\zeta$ *strongly (resp. compactly) for some* $\zeta < 0$*;*

(iii) $T_t^\alpha \to T_t$ *strongly (resp. compactly) for some* $t > 0$.

Moreover, if $\mathcal{E}_\alpha \to \mathcal{E}$ *compactly and the resolvents* R_ζ^α *are all compact, denoting by* λ_k *(respectively* λ_k^α*) the* h^{th} *eigenvalue of* L *(respectively of* L_α*) then*

$$\lim_\alpha \lambda_k^\alpha = \lambda_k$$

for any k.

3.2 Edge-like neighborhoods

In this section we address our study to some possible edge-neighborhoods, that might be patched together with vertex-neighborhoods in order to obtain a graph-like approximating structure for a general finite graph.

3.2.1 Cylindrical tubes around one edge.

Let $M \subset \mathbb{R}$ be an open interval (possibly unbounded) and $dm(x) := \rho(x)dx$, where dx is the 1-dimensional Lebesgue measure and $\rho : M \to (0, \infty)$ is a smooth density function. Denote by $B_2\left(0, \frac{1}{n}\right)$ the open ball centered in 0 and of radius $1/n$ from $\mathbb{R}^2$. We consider the tubes

$$M_n := M \times B_2\left(0, \frac{1}{n}\right) = \left\{(x, y, z) \in \mathbb{R}^3 : x \in M,\ y^2 + z^2 < \frac{1}{n^2}\right\}, n \in \mathbb{N}$$

with the measures $m_n(dxdydz) := \frac{n^2}{\pi}\rho(x)dxdydz$. Then $(M_n, m_n) \to (M, m)$ in the sense of the measured Gromov-Hausdorff convergence. Indeed, if we define f_n :

$M_n \to M$ by $f_n(x,y,z) = x$ for any $(x,y,z) \in M_n$ then for any $(x_1,y_1,z_1),(x_2,y_2,z_2) \in M_n$ we have either

$$|x_1 - x_2| \le \frac{1}{n} \Longrightarrow |d_n((x_1,y_1,z_1),(x_2,y_2,z_2)) - d(x_1,x_2)| \le$$

$$\le \sqrt{\frac{1}{n^2} + 2(y_1^2 + y_2^2 + z_1^2 + z_2^2)} + \frac{1}{n} \le \frac{1}{n}(1+\sqrt{5})$$

or

$$|x_1 - x_2| > \frac{1}{n} \Longrightarrow |d_n((x_1,y_1,z_1),(x_2,y_2,z_2)) - d(x_1,x_2)| =$$

$$\frac{|y_1-y_2|^2 + |z_1-z_2|^2}{\sqrt{|x_1-x_2|^2 + |y_1-y_2|^2 + |z_1-z_2|^2} + |x_1-x_2|} \le \frac{2(y_1^2+y_2^2+z_1^2+z_2^2)}{2|x_1-x_2|} < \frac{2}{n},$$

so in both cases $|d_n((x_1,y_1,z_1),(x_2,y_2,z_2)) - d(x_1,x_2)| < \frac{4}{n} \searrow 0$. The condition (3.1.5) is satisfied because

$$\frac{n^2}{\pi}\int_M \int_{B_2\left(0,\frac{1}{n}\right)} u \circ f_n(x,y,z)\,\rho(x)dxdydz = \int_M u(x)\,\rho(x)dx, \qquad \forall u \in C_0(M).$$

We consider now the following bilinear forms:

$$\mathcal{E}^n(u_n) := \frac{n^2}{\pi}\int_{M_n}\left(\left|\frac{\partial u_n}{\partial x}\right|^2 + \left|\frac{\partial u_n}{\partial y}\right|^2 + \left|\frac{\partial u_n}{\partial z}\right|^2\right)(x,y,z)\,\rho(x)dxdydz \qquad (3.2.1)$$

defined for any $u_n \in \mathcal{D}(\mathcal{E}^n) := H^1(M_n, m_n)$ where $H^1(M_n,m_n)$ is the $(1,2)$-Sobolev space included in $L_2(M_n,m_n)$. Define also

$$\mathcal{E}(u) := \int_M |u'(x)|^2 \rho(x)dx, \qquad \forall u \in \mathcal{D}(\mathcal{E}) := H^1(M,m). \qquad (3.2.2)$$

Then $(\mathcal{E}^n, \mathcal{D}(\mathcal{E}^n))$, $n \in \mathbb{N}$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ are Dirichlet forms respectively on $L_2(M_n,m_n)$, $n \in \mathbb{N}$ and $L_2(M,m)$.

Proposition 3.2.1. *The following assertions hold:*
(i) $\mathcal{E}^n \to \mathcal{E}$ *in the sense of* Γ*-convergence;*
(ii) The sequence $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ *is asymptotically compact.*

Proof. (i) In order to prove (3.1.8), we consider $\{u_n\}_{n\in\mathbb{N}}$ tending strongly to u, with $u_n \in L_2(M_n, m_n)$ and $u \in L_2(M, m)$ and suppose that $\liminf_{n\to\infty} \mathcal{E}^n(u_n) < \infty$. Replacing $\{u_n\}_{n\in\mathbb{N}}$ by a subsequence if necessary we may assume that $u_n \in \mathcal{D}(\mathcal{E}^n)$, $n \in \mathbb{N}$. The strong convergence $u_n \to u$ yields

$$\lim_{n\to\infty} \frac{n^2}{\pi} \int_{M_n} |u(x) - u_n(x,y,z)|^2 \rho(x) dx dy dz = 0$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{M_1} \left| u(x) - u_n\left(x, \frac{y}{n}, \frac{z}{n}\right) \right|^2 \rho(x) dx dy dz = 0 \tag{3.2.3}$$

Denote $v_n(x,y,z) := u_n\left(x, \frac{y}{n}, \frac{z}{n}\right)$ and $v(x,y,z) := u(x)$ for $(x,y,z) \in M_1$. The equality (3.2.3) shows that $v_n \to v$ in $L_2(M_1, m_1)$. Moreover, we have

$$\mathcal{E}^1(v_n) = \frac{1}{\pi} \int_{M_1} \left(\left| \frac{\partial u_n}{\partial x} \right|^2 + \frac{1}{n^2} \left| \frac{\partial u_n}{\partial y} \right|^2 + \frac{1}{n^2} \left| \frac{\partial u_n}{\partial z} \right|^2 \right) \left(x, \frac{y}{n}, \frac{z}{n}\right) \rho(x) dx dy dz \le$$

$$\frac{1}{\pi} \int_{M_1} \left(\left| \frac{\partial u_n}{\partial x} \right|^2 + \left| \frac{\partial u_n}{\partial y} \right|^2 + \left| \frac{\partial u_n}{\partial z} \right|^2 \right) \left(x, \frac{y}{n}, \frac{z}{n}\right) \rho(x) dx dy dz = \mathcal{E}^n(u_n), \ n \in \mathbb{N}.$$

Thus, $\sup_{n\in\mathbb{N}} \mathcal{E}^1(v_n) \le \sup_{n\in\mathbb{N}} \mathcal{E}^n(u_n) < \infty$. By using Lemma 2.12 from [MR92] we obtain $v \in H^1(M_1, m_1) = \mathcal{D}(\mathcal{E}^1)$ and $\mathcal{E}^1(v) \le \liminf_{n\to\infty} \mathcal{E}^1(v_n)$. Consequently $u \in H^1(M, m) = \mathcal{D}(\mathcal{E})$ and

$$\mathcal{E}(u) = \mathcal{E}^1(v) \le \liminf_{n\to\infty} \mathcal{E}^1(v_n) \le \liminf_{n\to\infty} \mathcal{E}^n(u_n),$$

which proves (3.1.8).

For the proof of (3.1.9), for each $u \in L_2(M, m)$ we choose $u_n := u \circ f_n$; obviously $u_n \to u$ strongly and $\mathcal{E}^n(u_n) = \mathcal{E}(u)$, $n \in \mathbb{N}$.

(ii) Let $\{u_n\}_{n\in N}$ be a sequence with $u_n \in L_2(M_n, m_n)$, $n \in \mathbb{N}$ such that $\sup_{n\in\mathbb{N}} \left(\mathcal{E}^n(u_n) + \|u_n\|^2_{L_2(M_n, m_n)} \right) < \infty$. We define

$$v_n(x) := \frac{n^2}{\pi} \int_{B_2\left(0, \frac{1}{n}\right)} u_n(x,y,z) \ dy dz, \ \forall x \in M, \ n \in \mathbb{N}.$$

Then

$$\|v_n\|^2_{L_2(M,m)} = \frac{n^4}{\pi^2} \int_M \left| \int_{B_2\left(0, \frac{1}{n}\right)} u_n(x,y,z) \ dy dz \right|^2 \rho(x) dx \le$$

$$\leq \frac{n^2}{\pi} \int_M \int_{B\left(0,\frac{1}{n}\right)} |u_n(x,y,z)|^2 \rho(x) dxdydz = \|u_n\|^2_{L_2(M_n,m_n)}$$

and similarly

$$\mathcal{E}(v_n) = \frac{n^4}{\pi^2} \int_M \left| \int_{B\left(0,\frac{1}{n}\right)} \frac{\partial u_n}{\partial x}(x,y,z) \ dydz \right|^2 \rho(x) dx \leq \mathcal{E}^n(u_n)$$

for any $n \in \mathbb{N}$, so the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded in the L_2-norm and in the seminorm $\mathcal{E}^{1/2}$ as well, and therefore there exists a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ L_2-convergent to some $v \in L_2(M,m)$. We shall prove in the sequel that $u_{n_k} \to v$ strongly, namely that

$$\lim_{k\to\infty} \frac{n_k^2}{\pi} \int_M \int_{B\left(0,\frac{1}{n_k}\right)} |u_{n_k}(x,y,z) - v_{n_k}(x)|^2 \rho(x) dxdydz = 0 \tag{3.2.4}$$

If we denote $w^x_{n_k}(y,z) := u_{n_k}\left(x, \frac{y}{n_k}, \frac{z}{n_k}\right)$ for $(y,z) \in B(0,1)$, $k \in \mathbb{N}$ and for fixed $x \in M$ then we have

$$\begin{aligned}
\frac{n_k^2}{\pi} \int_{B\left(0,\frac{1}{n_k}\right)} & \left| u_{n_k}(x,y,z) - \frac{n_k^2}{\pi} \int_{B\left(0,\frac{1}{n_k}\right)} u_n(x,y,z) \ dydz \right|^2 \ dydz \\
= \ & \frac{1}{\pi} \int_{B(0,1)} \left| w^x_{n_k}(y,z) - \frac{1}{\pi} \int_{B(0,1)} w^x_{n_k}(y,z) \ dydz \right|^2 \ dydz \\
\leq \ & \frac{C}{\pi} \mathcal{E}_{B(0,1)}\left(w^x_{n_k}\right) = \frac{C}{\pi} \int_{B(0,1)} \left(\left| \frac{\partial w^x_{n_k}}{\partial y} \right|^2 + \left| \frac{\partial w^x_{n_k}}{\partial z} \right|^2 \right)(y,z) \ dydz \\
= \ & \frac{C}{\pi} \int_{B(0,1)} \frac{1}{n_k^2} \left(\left| \frac{\partial u_{n_k}}{\partial y} \right|^2 + \left| \frac{\partial u_{n_k}}{\partial z} \right|^2 \right) \left(\frac{y}{n_k}, \frac{z}{n_k} \right) \ dydz \\
= \ & \frac{C}{n_k} \int_{B\left(0,\frac{1}{n_k}\right)} \left(\left| \frac{\partial u_{n_k}}{\partial y} \right|^2 + \left| \frac{\partial u_{n_k}}{\partial z} \right|^2 \right)(y,z) \ dydz
\end{aligned}$$

where we have applied the Poincaré inequality for the classical Dirichlet form $(\mathcal{E}_{B(0,1)}, H^1(B(0,1)))$ given by

$$\mathcal{E}_{B(0,1)}(w) = \int_{B(0,1)} \left(\left| \frac{\partial w}{\partial y} \right|^2 + \left| \frac{\partial w}{\partial z} \right|^2 \right)(y,z) \ dydz, \ w \in H^1(B(0,1)).$$

Consequently,

$$\lim_{k\to\infty} \frac{n_k^2}{\pi} \int_M \int_{B\left(0,\frac{1}{n_k}\right)} |u_{n_k}(x,y,z) - v_{n_k}(x)|^2 \rho(x) dxdydz \leq \frac{C}{\pi} \lim_{k\to\infty} \frac{1}{n_k^2} \mathcal{E}_{n_k}(u_{n_k}) = 0,$$

because $\sup_{k\in\mathbb{N}} \mathcal{E}_{n_k}(u_{n_k}) < \infty$. This shows the strong convergence of $\{u_{n_k}\}_k$ to v and ends the proof of the asymptotic compactness. □

Corollary 3.2.2. *The sequence $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ compactly converges to $\mathcal{E}$.*

Proof. The assertion follows from Proposition 3.2.1 and Theorem 3.1.10. □

The unique self-adjoint and non-negative operator L associated with the Dirichlet form $\mathcal{E}$ is given by

$$Lu = -\frac{1}{\rho}(\rho u')' \text{ for } u \in H^2(M, m).$$

Denote by L_n the infinitesimal generators associated with the Dirichlet forms $\mathcal{E}^n$. According to Theorem 3.1.10 and Corollary 3.2.2, for the corresponding strongly continuous contraction semigroups and the strongly continuous resolvents we have:

Corollary 3.2.3. *(i) $R^n_\zeta \to R_\zeta$ compactly for some $\zeta < 0$;*
(ii) $T^n_t \to T_t$ compactly for some $t > 0$.

A standard Rellich embedding theorem asserts that if $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ then the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact (see for instance [Am78] or [Ne67]). The Rellich compact embedding theorem gives the compactness of the resolvent. Theorem 3.1.10 allows us to state the convergence of spectra, under a boundedness assumption:

Corollary 3.2.4. *If M is a bounded interval of $\mathbb{R}$ and the density function ρ is bounded then every kth eigenvalue of L_n converges to the corresponding kth eigenvalue of L.*

Remark 3.2.5. One can prove a result similar to Proposition 3.2.1 for approximations by "empty tubes" $\widehat{M_n} := M \times S_2(0, 1/n)$ equipped with the metric g_n induced by the euclidian metric of $\mathbb{R}^3$, written in local coordinates (t, φ_n) as

$$g_n := dt^2 + \frac{1}{n^2} d\varphi_n{}^2.$$

The measures $\widehat{m}_n$ will be supposed to have $\frac{n}{2\pi}\rho(t)$ as weight w.r.t the Lebesgue measure. Equivalently, one can work on M_1 with the metric

$$g_n := dt^2 + \frac{1}{2\pi n^2}\rho(t) d\varphi_1{}^2$$

and with the Lebesgue measure.

Defining now the Dirichlet form

$$\widehat{\mathcal{E}}^n(u_n) := \int_{\widehat{M}_n} |\nabla u_n|^2 d\widehat{m}_n,\ u_n \in H^1(\widehat{M}_n, \widehat{m}_n)$$

we have with a proof absolutely similar to the one of Proposition 3.2.1 that:

(i) $\widehat{\mathcal{E}}^n \to \mathcal{E}$ in the sense of Γ-convergence;

(ii) The sequence $\{\widehat{\mathcal{E}}^n\}_{n\in\mathbb{N}}$ is asymptotically compact.

Again, this compact convergence of the Dirichlet forms will have as a consequence the convergence of the corresponding resolvents and semigroups.

For bounded M, by mapping the surface $\widehat{M}_n$ into the chart consisting of an annulus in $\mathbb{R}^2$ we obtain a Rellich compact embedding of the Sobolev space $H^1(\widehat{M}_n, \widehat{m}_n)$ into $L_2(\widehat{M}_n, \widehat{m}_n)$. That leads to the compactness of the resolvent and eventually to the convergence of spectra, by Theorem 3.1.10.

In fact, we can prove a result similar to Proposition 3.2.1 with some boundedness assumptions and with a more general density function ρ. Consider now the bounded interval M, but with a measure m that has a density function $\rho : M \to [\alpha, \beta] \subset (0, \infty)$ with respect to dx, which has a discontinuity point $x_0 \in M =: (a, b)$ such that $\rho|_{(a,x_0)}$ and $\rho|_{(x_0,b)}$ are smooth and both limits $\rho_-(x_0) := \lim_{x\uparrow x_0} \rho(x)$ and $\rho_+(x_0) := \lim_{x\downarrow x_0} \rho(x)$ exist and are finite. Consider also smooth functions $\rho_n : M \to [\alpha, \beta]$ so that $\rho_n = \rho$ on $M \setminus (x_0 - \epsilon/n, x_0 + \epsilon/n)$ and define $m_n(dx) := \rho_n(x)dx$ as a measure on M_n. Put

$$\mathcal{E}^n(u_n) := \frac{n^2}{\pi} \int_{M_n} \left(\left|\frac{\partial u_n}{\partial x}\right|^2 + \left|\frac{\partial u_n}{\partial y}\right|^2 + \left|\frac{\partial u_n}{\partial z}\right|^2 \right) (x, y, z)\, \rho_n(x) dxdydz$$

for any $u_n \in \mathcal{D}(\mathcal{E}^n) := H^1(M_n, m_n)$ and

$$\mathcal{E}(u) := \int_{(a,x_0)} |u'(x)|^2 \rho(x)dx + \int_{(x_0,b)} |u'(x)|^2 \rho(x)dx,$$

for all $u \in \mathcal{D}(\mathcal{E}) := \{u \in C(M) : u|_{(a,x_0)} \in H^1((a, x_0), m), u|_{(x_0,b)} \in H^1((x_0, b), m)\}$.

We are dealing in this way with a sort of singularity in x_0, that will be a basic model for patching edges together in order to construct a graph.

Proposition 3.2.6. *The following assertions hold:*

(i) $\mathcal{E}^n \to \mathcal{E}$ in the sense of Γ-convergence;

(ii) The sequence $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ is asymptotically compact.

Proof. The proof follows essentially the arguments from the proof of Proposition 3.2.1. Now, instead of reducing the calculus on (M_1, m_1) we can use M_1 equipped with the Lebesgue measure. For Γ-convergence we construct again v and v_n and show that $v_n \to v$ in $L_2(M_1, dxdydz)$ and $v \in H^1(M_1, dxdydz)$, that gives $u|_{(a,x_0)} \in H^1((a, x_0), m), u|_{(x_0,b)} \in H^1((x_0, b), m)\}$ together with the continuity of u in x_0. Essential for following the steps of the proof of Proposition 3.2.1 is the hypothesis $0 < \alpha \le \rho \le \beta$ on M, that will permit to reduce integrals with respect to m_n to integrals with respect to the Lebesgue measure. The proof of asymptotic compactness goes exactly on the same lines. □

Let us observe that the associated "Laplacian" on M is an operator which acts as
$$Lu = -\frac{1}{\rho}(\rho u')'$$
on each interval (a, x_0) and (x_0, b) for any $u \in H^2((a, x_0), \rho dx) \bigoplus H^2((x_0, b), \rho dx)$, u continuous in x_0. Moreover, all elements u from the domain of L must satisfy the condition
$$\rho_-(x_0)u'_-(x_0) + \rho_+(x_0)u'_+(x_0) = 0$$
which is already a Kirchhoff boundary condition in x_0.

3.2.2 Weighted tubes with variable width

Let M be an open interval of $\mathbb{R}$ endowed with the 1-dimensional Lebesgue measure $m = dx$. We consider two continuous functions $\alpha_n, \beta_n : M \to \mathbb{R}_+$, $n \in \mathbb{N}$ and set
$$M_n := \left\{(x, y, z) \in \mathbb{R} : x \in M,\ y^2 + z^2 < \alpha_n(x)^2\right\},\ n \in \mathbb{N}.$$

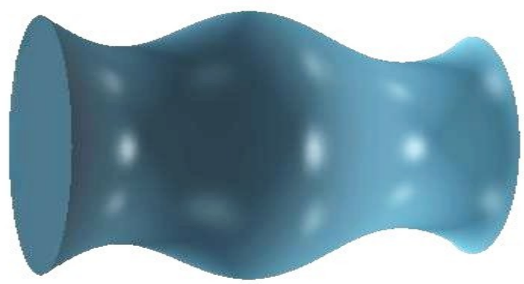

Figure 3.1

We endow the sets M_n with the measures m_n defined as
$$m_n(dxdydz) := \frac{\beta_n(x)}{\pi}dxdydz,\ n \in \mathbb{N}.$$
The L_2-norm of an element $u_n \in L_2(M_n, m_n)$ will be
$$\|u_n\|_{L_2(M_n,m_n)} = \left(\frac{1}{\pi}\int_M \beta_n(x)\int_{B_2(0,\alpha_n(x))} |u_n(x,y,z)|^2\ dydzdx\right)^{1/2},\ n \in \mathbb{N}.$$

Proposition 3.2.7. *If $\alpha_n \searrow 0$ uniformly on M and $\alpha_n^2\beta_n \to 1$ uniformly on M then $(M_n, m_n) \to (M, m)$ in the sense of the measured Gromov-Hausdorff convergence.*

Proof. The first two properties requested by the measured Gromov-Hausdorff convergence are satisfied because the uniform convergence $\alpha_n \searrow 0$ implies $\sup_M \alpha_n \searrow 0$ and if we define again $f_n : M_n \to M$ by $f_n(x,y,z) = x$ then we get

$$|d_n((x_1,y_1,z_1),(x_2,y_2,z_2)) - d(x_1,x_2)| < 4 \sup_M \alpha_n$$

where $(x_1,y_1,z_1),(x_2,y_2,z_2) \in M_n$.

If $u \in C_0(M)$ then

$$\frac{1}{\pi}\int_{M_n} u \circ f_n(x,y,z)\,\beta_n(x)\ dxdydz = \int_M u(x)\,\beta_n(x)\,\alpha_n(x)^2\ dx \to \int_M u(x)\ dx,$$

which gives the property (3.1.5). □

We define in a similar way the Dirichlet forms $(\mathcal{E}^n, \mathcal{D}(\mathcal{E}^n))$, $n \in \mathbb{N}$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ by setting $\mathcal{D}(\mathcal{E}^n) := H^1(M_n, m_n)$, $\mathcal{D}(\mathcal{E}) := H^1(M,m)$ and

$$\begin{aligned}
\mathcal{E}^n(u_n) &:= \frac{1}{\pi}\int_M \beta_n(x)\int_{B(0,\alpha_n(x))}\left(\left|\frac{\partial u_n}{\partial x}\right|^2 + \left|\frac{\partial u_n}{\partial y}\right|^2 + \left|\frac{\partial u_n}{\partial z}\right|^2\right)(x,y,z)\ dxdydz \\
\mathcal{E}(u) &:= \int_M |u'(x)|^2\,dx
\end{aligned}$$

for $u_n \in H^1(M_n, m_n)$, $n \in \mathbb{N}$ and $u \in H^1(M,m)$ respectively.

We suppose that $\alpha_n \searrow 0$ and $\alpha_n^2\beta_n \to 1$, both uniformly on M.

Proposition 3.2.8. *Let us assume in addition that $\alpha_n \in C^1(M)$. Then the following assertions hold:*

(i) If $\alpha_n' \to 0$ uniformly on M and $\{\beta_n(x)\}_{n\in\mathbb{N}}$ uniformly bounded from below by a positive constant then $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ is Γ-convergent to $\mathcal{E}$.

(ii) If the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ is uniformly bounded then $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ is asymptotically compact.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in L_2(M_n, m_n)$, $n \in \mathbb{N}$ be a sequence strongly convergent to $u \in L_2(M,m)$ which implies

$$\lim_{n\to\infty}\int_M \beta_n(x)\int_{B(0,\alpha_n(x))}|u(x) - u_n(x,y,z)|^2\ dydzdx = 0$$

or equivalently

$$\lim_{n\to\infty}\int_M \beta_n(x)\int_{B(0,1)}|u(x) - u_n(x,\alpha_n(x)\,y,\alpha_n(x)\,z)|^2\ dydzdx = 0 \qquad (3.2.5)$$

Denote $v_n(x,y,z) := u_n(x,\alpha_n(x)y,\alpha_n(x)z)$ and $v(x,y,z) := u(x)$ for each (x,y,z) in $\widetilde{M}_1 := M \times B(0,1)$. By our hypothesis there exists a constant $c > 0$ such that $\beta_n(x) \geq c$ for each $x \in M$ and $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \int_{\widetilde{M}_1} |v(x,y,z) - v_n(x,y,z)|^2 \; dxdydz$$

$$\leq \frac{1}{c} \lim_{n\to\infty} \int_M \beta_n(x) \int_{B(0,1)} |u(x) - u_n(x,\alpha_n(x)y,\alpha_n(x)z)|^2 \; dydzdx = 0$$

and thus $v_n \to v$ in $L_2(\widetilde{M}_1, dxdydz)$.

In order to prove the first property (3.1.8) from the definition of Γ-convergence we may suppose $\liminf_{n\to\infty} \mathcal{E}^n(u_n) < \infty$. In fact, replacing $\{u_n\}_{n\in\mathbb{N}}$ by a subsequence if necessary we may suppose that $u_n \in H^1(M_n, m_n) = \mathcal{D}(\mathcal{E}^n)$ for each $n \in \mathbb{N}$. Then $v_n \in H^1(\widetilde{M}_1)$ and

$$\begin{aligned}
\mathcal{E}_{\widetilde{M}_1}(v_n) &= \frac{1}{\pi} \int_{\widetilde{M}_1} |\nabla v_n|^2 (x,y,z)\, dxdydz \\
&= \frac{1}{\pi} \int_{\widetilde{M}_1} \Bigg(\left| \frac{\partial u_n}{\partial x} + \alpha_n'(x) \frac{\partial u_n}{\partial y} + \alpha_n'(x) \frac{\partial u_n}{\partial z} \right|^2 + \alpha_n^2(x) \left| \frac{\partial u_n}{\partial y} \right|^2 \\
&\qquad + \alpha_n^2(x) \left| \frac{\partial u_n}{\partial z} \right|^2 \Bigg) (x,\alpha_n(x)y,\alpha_n(x)z) \; dxdydz.
\end{aligned}$$

By using the inequality $ab + bc + ca \leq a^2 + b^2 + c^2$, $a,b,c \in \mathbb{R}$ we obtain

$$\frac{\partial u_n}{\partial x}\frac{\partial u_n}{\partial y} + \frac{\partial u_n}{\partial y}\frac{\partial u_n}{\partial z} + \frac{\partial u_n}{\partial z}\frac{\partial u_n}{\partial x} \leq |\nabla u_n|^2 . \tag{3.2.6}$$

.

Let $0 < \varepsilon < 1$ be fixed. By using our hypothesis there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $x \in M$ we have simultaneously

$$\begin{aligned}
\alpha_n(x) &< 1 \\
\alpha_n'(x) &< \varepsilon \\
\alpha_n(x)^2 \beta_n(x) &> 1 - \varepsilon,
\end{aligned}$$

which, together with (3.2.6), yields

$$\begin{aligned}
\mathcal{E}_{\widetilde{M}_1}(v_n) &\leq \frac{1}{\pi}(1+3\varepsilon) \int_{\widetilde{M}_1} \frac{\alpha_n(x)^2 \beta_n(x)}{1-\varepsilon} |\nabla u_n|^2 (x,\alpha_n(x)y,\alpha_n(x)z) \; dxdydz \\
&= \frac{1+3\varepsilon}{1-\varepsilon} \mathcal{E}^n(u_n),
\end{aligned}$$

for any $n \geq n_0$. Then

$$\limsup_{n\to\infty} \mathcal{E}_{\widetilde{M_1}}(v_n) \leq \frac{1+3\varepsilon}{1-\varepsilon} \liminf_{n\to\infty} \mathcal{E}^n(u_n)$$

for each $0 < \varepsilon < 1$. Now if we let ε to tend to 0 we get

$$\limsup_{n\to\infty} \mathcal{E}_{\widetilde{M_1}}(v_n) \leq \liminf_{n\to\infty} \mathcal{E}^n(u_n) < \infty.$$

Since we have also proved $v_n \to v$ in $L_2(\widetilde{M_1})$, according to Lemma 2.12 in [MR92] one obtains $v \in \mathcal{D}(\mathcal{E}_{\widetilde{M_1}}) = H^1(\widetilde{M_1})$ and $\mathcal{E}_{\widetilde{M_1}}(v) \leq \liminf_{n\to\infty} \mathcal{E}_{\widetilde{M_1}}(v_n)$. Finally we obtain $u \in H^1(M_n, m_n)$ and

$$\mathcal{E}(u) = \mathcal{E}_{\widetilde{M_1}}(v) \leq \liminf_{n\to\infty} \mathcal{E}_{\widetilde{M_1}}(v_n) \leq \liminf_{n\to\infty} \mathcal{E}^n(u_n),$$

which proves (3.1.8).

In order to prove (3.1.9), for each $u \in L_2(M,m)$ we choose $u_n = u \circ f_n$. Obviously $u_n \to u$ strongly and $\mathcal{E}^n(u_n) \to \mathcal{E}(u)$.

(ii) Let now $\{u_n\}_{n\in\mathbb{N}}$ be a sequence with $u_n \in L_2(M_n, m_n)$, $n \in \mathbb{N}$, such that

$$\sup_{n\in\mathbb{N}} (\mathcal{E}^n(u_n) + \|u_n\|^2_{L_2(M_n,m_n)}) < \infty.$$

For $x \in M$ arbitrarily fixed we define

$$\begin{aligned} v_n(x) &:= \frac{1}{\pi\alpha_n(x)^2} \int_{B(0,\alpha_n(x))} u_n(x,y,z) \\ &= \frac{1}{\pi} \int_{B(0,1)} u_n(x, \alpha_n(x) y, \alpha_n(x) z)\ dydz \end{aligned}$$

Then

$$\begin{aligned} \|v_n\|^2_{L_2(M,m)} &= \frac{1}{\pi^2} \int_M \left| \int_{B(0,1)} u_n(x, \alpha_n(x) y, \alpha_n(x) z)\ dydz \right|^2 dx \\ &\leq k_1 \|u_n\|^2_{L_2(M_n,m_n)} \end{aligned}$$

with $k_1 > 0$ constant, and

$$\begin{aligned} &\mathcal{E}(v_n) = \int_M |v_n(x)|^2\ dx \\ = &\ \frac{1}{\pi^2} \int_M \left| \int_{B(0,1)} \left(\frac{\partial u_n}{\partial x} + \alpha_n'(x) \frac{\partial u_n}{\partial y} + \alpha_n'(x) \frac{\partial u_n}{\partial z} \right)(x, \alpha_n(x) y, \alpha_n(x) z)\, dydz \right|^2 dx \\ \leq &\ k_2 \mathcal{E}^n(u_n), \end{aligned}$$

since $\{\alpha_n'\}_{n\in\mathbb{N}}$ is uniformly bounded by our hypothesis. We proved that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in the L_2-norm and in the $\mathcal{E}^{1/2}$-seminorm, which implies the existence of a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ L_2-convergent to a $v \in L_2(M,m)$. We will show that $u_{n_k} \to v$ strongly, more precisely we will prove

$$\lim_{k\to\infty} \int_M \beta_{n_k}(x) \int_{B\left(0,\alpha_{n_k}(x)\right)} |u_{n_k}(x,y,z) - v_{n_k}(x)|^2 \; dydzdx. \tag{3.2.7}$$

For each $x \in M$ fixed we define $w_{n_k}^x(y,z) := u_{n_k}(x, \alpha_{n_k}(x)y, \alpha_{n_k}(x)z)$ for $(y,z) \in B(0,1)$, $k \in \mathbb{N}$. Then, by applying the Poincaré inequality for the classical Dirichlet form $(\mathcal{E}_{B(0,1)}, H^1(B(0,1))$ we get

$$\begin{aligned}
\int_{B\left(0,\alpha_{n_k}(x)\right)} & |u_{n_k}(x,y,z) - v_{n_k}(x)| \; dydz \\
&= \alpha_{n_k}(x)^2 \int_{B(0,1)} \left| w_{n_k}^x(y,z) - \frac{1}{\pi}\int_{B(0,1)} w_{n_k}^x(y,z)\, dydz \right|^2 dydz \\
&\leq C\alpha_{n_k}(x)^2 \mathcal{E}_{B(0,1)}\left(w_{n_k}^x\right) = C\alpha_{n_k}(x)^2 \int_{B(0,1)} \left|\nabla w_{n_k}^x\right|^2 (y,z)\, dydz \\
&\leq C\alpha_{n_k}(x)^4 \int_{B(0,1)} |\nabla u_{n_k}|^2 (x, \alpha_{n_k}(x)y, \alpha_{n_k}(x)z)\, dydz
\end{aligned}$$

for any $x \in M$ fixed. Therefore we obtain

$$\begin{gathered}
\lim_{k\to\infty} \int_M \beta_{n_k}(x) \int_{B\left(0,\alpha_{n_k}(x)\right)} |u_{n_k}(x,y,z) - v_{n_k}(x)|^2 \; dydzdx \\
\leq C \lim_{k\to\infty} \int_M \beta_{n_k}(x)\, \alpha_{n_k}(x)^2 \int_{B\left(0,\alpha_{n_k}(x)\right)} |\nabla u_{n_k}|^2 (x,y,z)\, dydzdx \\
\leq C \lim_{k\to\infty} \|\alpha_{n_k}\|_{L^\infty(M,m)} \mathcal{E}^{n_k}(u_{n_k}) = 0
\end{gathered}$$

because $\alpha_{n_k} \to 0$ uniformly and $\sup_{k\in\mathbb{N}} \mathcal{E}^{n_k}(u_{n_k}) < \infty$ and then $u_{n_k} \to v$ strongly, which ends the proof. □

Corollary 3.2.9. *If $\{\alpha_n\}_{n\in\mathbb{N}} \subset C^1(M)$, $\{\beta_n\}_{n\in\mathbb{N}} \subset C(M)$ with $\alpha_n > 0$, $\beta_n \geq k > 0$ on M, $n \in N$ such that $\alpha_n \searrow 0$, $\alpha_n' \to 0$, $\alpha_n^2\beta_n \to 1$ uniformly on M then*

(i)$\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ compactly converges to $\mathcal{E}$.

(ii) $R_\zeta^n \to R_\zeta$ compactly for some $\zeta < 0$;

(iii) $T_t^n \to T_t$ compactly for some $t > 0$.

Moreover, if M is a bounded interval of $\mathbb{R}$ then every kth eigenvalue of L_n converges to the corresponding kth eigenvalue of L.

Remark 3.2.10. One can prove similar results for approximations by surfaces

$$\widehat{M_n} := \left\{(x,y,z) \in \mathbb{R} : x \in M,\ y^2 + z^2 = \alpha_n (x)^2\right\},\ n \in \mathbb{N}.$$

equipped with metric g_n induced by the euclidian metric of $\mathbb{R}^3$, written in local coordinates (t, φ_n) as

$$g_n := dt^2 + \alpha_n (t)^2 \, d{\varphi_n}^2.$$

The measures $\widehat{m}_n$ will have $\beta_n(t)/2\pi$ as weight w.r.t the Lebesgue measure.

Defining now the Dirichlet form

$$\widehat{\mathcal{E}}^n(u_n) := \int_{\widehat{M_n}} |\nabla u_n|^2 d\widehat{m}_n,\ u_n \in H^1(\widehat{M_n}, \widehat{m}_n),$$

under the assumptions that $\{\alpha_n\}_{n\in\mathbb{N}} \subset C^1(M)$, $\{\beta_n\}_{n\in\mathbb{N}} \subset C(M)$ with $\alpha_n > 0$, $\beta_n \geq k > 0$ on M, $n \in N$ such that $\alpha_n \searrow 0$, $\alpha_n' \to 0$, $\alpha_n^2\beta_n \to 1$ uniformly on M we have

(i) $\widehat{\mathcal{E}}^n \to \mathcal{E}$ in the sense of Γ-convergence;

(ii) The sequence $\{\widehat{\mathcal{E}}^n\}_{n\in\mathbb{N}}$ is asymptotically compact.

(iii) $\widehat{R}^n_\zeta \to R_\zeta$ compactly for some $\zeta < 0$;

(iv) $\widehat{T}^n_t \to T_t$ compactly for some $t > 0$.

3.3 The case of the N-spider

We consider now the simplest example of graph, whose study will serve for defining graph-like structures approximating more general finite graphs.

Let $N \geq 3$ be a finite integer number. We consider the graph $M := \left(\bigcup_{i=1}^N e_i\right) \cup \{O\}$ consisting of N edges emanating from the vertex O. An edge e_i can be identified with an interval $I_i = [0, l_i]$ where l_i is the length of e_i. One can see M as a metric measure space with the graph metric and with the measure m such that m is given by $\rho_i(x)dx$ on each edge e_i, with $\rho_i : I_i \to (0,\infty)$ a smooth density function for any $i = 1, \dots, N$. We consider

$$L_2(M,m) = \bigoplus_{i=1}^N L_2(I_i, \rho_i(x)dx)$$

and

$$\|u\|_M^2 = \sum_{i=1}^N \|u_i\|_{I_i}^2 = \sum_{i=1}^N \int_{I_i} |u_i(x)|^2 \rho_i(x)\ dx,$$

where $u_i := u|_{e_i} \in L_2(I_i, \rho_i(x)dx)$.

We consider also the open sets $M_n \supset \left\{x \in \mathbb{R}^3 : d\,(x, M) < \frac{1}{n}\right\}$ with the euclidian metric and such that there exists a sequence $r_n \searrow 0$ with

$$M_n \setminus B_3\,(O, r_n) = \left\{x \in \mathbb{R}^3 : d\,(x, M) < \frac{1}{n}\right\} \setminus B_3\,(O, r_n)\,.$$

Additionally we shall assume in the sequel that r_n^3 goes to 0 faster than $1/n^2$:

$$n^2 r_n^3 \to 0 \text{ as } n \to \infty. \tag{3.3.1}$$

This assumption ensures the fact that the vertex-neighborhood part of M_n enclosed in the ball $B_3\,(O, r_n)$ shrinks to the vertex O faster than the edge-neighborhood part shrinks to the union of edges. Notice that the edge-neighborhood part of M_n consists of the union of N cylinders around interior segments of the edges. We decompose $M_n = M_n^{tube} \cup (M_n \setminus M_n^{tube})$ where $M_n^{tube} = \bigcup_{i=1}^N M_{n,i}^{tube}$ is the union of those N cylindrical tubes of radius $1/n$ contained in M_n.

Let us equip the metric space M_n with a measure m_n which is absolutely continuous with respect to the 3-dimensional Lebesgue measure λ^3:

$$m_n := \frac{n^2}{\pi}\theta_n \cdot \lambda^3|_{M_n}.$$

We choose the corresponding density $\theta_n : M_n \to (0, \infty)$ to be smooth and such that $(M_{n,i}^{tube}, m_n|_{M_{n,i}^{tube}})$ is isomorphic with $(I_i \times B_2(0, 1/n), \frac{n^2}{\pi}\rho_i(x)dxdydz)$ as metric measure spaces. Suppose also that $0 < a \le \theta_n \le b$ uniformly on M_n.

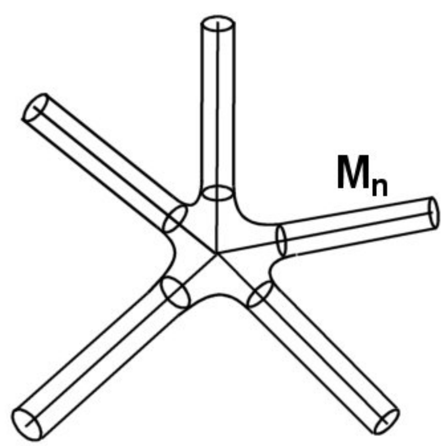

Figure 3.2

Proposition 3.3.1. *$(M_n, m_n) \to (M, m)$ in the sense of measured Gromov-Hausdorff convergence.*

Proof. We denote by $M_{n,i}$ the full tube around the edge e_i, $i = 1, ..., N$ and by $f_{n,i} : M_{n,i} \to I_i$, $i = 1, ..., N$ the projection that we used in subsection 3.2.1. We define $f_n : M_n \to M$ the continuous projection of the whole set M_n on the graph M such that $f_n = f_{n,i}$ on $M_{n,i}^{tube}$, $i = 1, ..., N$.

For $u \in C_0(M)$ we have

$$\lim_{n\to\infty} \int_{M_n} u \circ f_n \; dm_n = \lim_{n\to\infty} \int_{M_n^{tube}} u \circ f_n \; dm_n + \lim_{n\to\infty} \int_{M_n \setminus M_n^{tube}} u \circ f_n \; dm_n =$$

$$\begin{aligned} = & \sum_{1=1}^{N} \lim_{n\to\infty} \int_{M_{n,i}} u \circ f_{n,i} \; dm_n + \lim_{n\to\infty} \int_{M_n \setminus M_n^{tube}} u \circ f_n \; dm_n \\ - & \sum_{i=1}^{N} \lim_{n\to\infty} \int_{M_{n,i} \setminus M_{n,i}^{tube}} u \circ f_{n,i} \; dm_n, \end{aligned}$$

but

$$\begin{aligned} \int_{M_n \setminus M_n^{tube}} u \circ f_n dm_n & \leq \|u\|_{L^\infty} m_n(M_n \setminus M_n^{tube}) \leq \|u\|_{L^\infty} m_n(B(O, r_n)) \\ & = \|u\|_{L^\infty} \frac{n^2}{\pi} \frac{4\pi r_n^3}{3} \sup_{B_3(O,r_n)} \theta_n \to 0 \text{ as } n \to \infty \end{aligned}$$

and similarly

$$\int_{M_{n,i} \setminus M_{n,i}^{tube}} u \circ f_{n,i} \; dm_n \to 0 \text{ as } n \to \infty.$$

Therefore, by the measured Gromov-Hausdorff convergence that we have got for tubes within 3.2.1, we have

$$\lim_{n\to\infty} \int_{M_n} u \circ f_n \; dm_n = \sum_{i=1}^{N} \int_{e_i} u \; dm = \int_M u \; dm, \; u \in C_0(M).$$

□

We shall work in the sequel with stronger assumption on the approximating sets M_n, namely that *our ε_n-approximations $f_n : M_n \to M$ may be chosen to be Lipschitz*. See that at least for $M_n = \left\{x \in \mathbb{R}^3 : d(x, M) < \frac{1}{n}\right\}$ the maps f_n can be chosen Lipschitz.

Let us define the Dirichlet forms

$$\mathcal{E}(u) := \sum_{i=1}^{N} \int_{I_i} |u'(x)|^2 \rho_i(x) \; dx, \text{ for } u \in C(M) \text{ with } u|_{I_i} \in H^1(I_i), \; \forall i \quad (3.3.2)$$

$$\mathcal{E}^n(u_n) := \int_{M_n} |\nabla u_n|^2 \; dm_n, \text{ for } u_n \in H^1(M_n), \; n \in \mathbb{N}. \tag{3.3.3}$$

Then we have the following convergence result

Theorem 3.3.2. *(i) $\mathcal{E}^n \to \mathcal{E}$ in the sense of Γ-convergence.*
(ii) The sequence $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ is asymptotically compact.

Proof. (i) Let us consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in L_2(M_n, m_n)$ and $u \in L_2(M, m)$ such that $u_n \to u$ strongly. We have to prove that

$$\mathcal{E}(u) \leq \liminf_{n\to\infty} \mathcal{E}^n(u_n). \tag{3.3.4}$$

We may suppose $\liminf_{n\to\infty} \mathcal{E}^n(u_n) < \infty$; replacing $\{u_n\}_{n\in\mathbb{N}}$ by a subsequence if necessary we may assume $u_n \in \mathcal{D}(\mathcal{E}^n)$, $n \in \mathbb{N}$. We decompose $M_n = M_n^\delta \cup (M \setminus M_n^\delta)$, $M = M^\delta \cup (M \setminus M^\delta)$ where $M_n^\delta := M_n \setminus B(0,\delta)$, $M := M \setminus B(0,\delta)$ for $\delta > 0$ arbitrarily fixed. Denote

$$\begin{aligned} \mathcal{E}_\delta^n(v_n) \; : &= \int_{M_n^\delta} |\nabla v_n|^2 \; dm_n, \qquad v_n \in H^1(M_n^\delta) \\ \mathcal{E}_\delta(v) \; : &= \int_{M^\delta} |\nabla v|^2 \; dm, \qquad v \in H^1(M^\delta) \end{aligned}$$

For $\delta > 0$ fixed and n large enough M_n^δ is the disjoint union of N cylinders for which we have proved already the Γ-convergence $\mathcal{E}_\delta^n \to \mathcal{E}_\delta$. Since $u_n \to u$ implies $u_n|_{M_n^\delta} \to u|_{M^\delta}$ we conclude that

$$\mathcal{E}_\delta(u|_{M^\delta}) \leq \liminf_{n\to\infty} \mathcal{E}_\delta^n\left(u_n|_{M_n^\delta}\right) \leq \liminf_{n\to\infty} \mathcal{E}^n(u_n), \qquad \delta > 0. \tag{3.3.5}$$

Obviously $u|_{I_i} \in H^1(I_i, dx)$ for each $i = 1, \ldots, N$ because $u \in H^1(M^\delta)$ for each $\delta > 0$. If we knew that u lies in $C(M)$ then

$$\mathcal{E}(u) = \mathcal{E}_\delta(u|_{M^\delta}) + \sum_{i=1}^N \int_{I_i \setminus M^\delta} |u'(x)|^2 \, dx$$

and the last term tends to 0 for $\delta \to 0$, which together with (3.3.5) yields (3.3.4). Therefore it remains to prove $u \in C(M)$.

Let us consider a set $\widehat{M_n} \subset M_n$with $\widehat{M_n} \cap M_{n,i}^{tube} = \emptyset$ for $i = 3, ..., N$, with $M_{n,1}^{tube}, M_{n,2}^{tube} \subset \widehat{M_n}$ and such that there exist the maps $\Psi_n : J_n \to \widehat{M_n}$ with J_n cylindrical tube of radius $1/n$ around the segment J, Ψ_n bijection with $\Psi_n \in C^1(J_n)$, $\Psi_n^{-1} \in C^1(\widehat{M_n})$ and Ψ_n bi-Lipschitz. We identify the segment J with $e_1 \cup e_2 \cup \{O\}$ by the continuous bi-Lipschitz map $\Psi : J \to e_1 \cup e_2 \cup \{O\}$. We consider the projections $\widehat{f_n} : \widehat{M_n} \to e_1 \cup e_2 \cup \{O\}$, $\widehat{f_n} := \Psi \circ \varphi_n \circ \Psi_n^{-1}$ where $\varphi_n : J_n \to J$ the projections that we used for cylindrical tubes in 3.2.1. Because $M_{n,1}^{tube}, M_{n,2}^{tube} \subset \widehat{M_n}$ are cylindrical

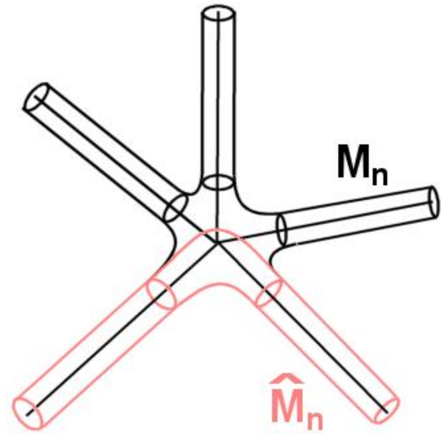

Figure 3.3

tubes of radius $\frac{1}{n}$, one has $\widehat{f}_n = f_n$ on $M_{n,1}^{tube} \cup M_{n,2}^{tube}$. Let m'_n be the measure on J_n given by $m'_n(A) := m_n(\Psi_n(A))$ for any Borel set $A \subset J_n$. Denote by m' the measure obtained in a similar way on J: $m' = m \circ \Psi$.

We shall prove that $u_n|_{\widehat{M_n}} \to u|_{e_1 \cup e_2 \cup \{O\}}$ strongly for the measured Gromov-Hausdorff convergence $(\widehat{M_n}, m_n|_{\widehat{M_n}}) \to \left(e_1 \cup e_2 \cup \{O\}, m|_{e_1 \cup e_2 \cup \{O\}}\right)$ with ε_n-approximations $\widehat{f}_n$. Since $u_n \to u$ strongly for the measured Gromov-Hausdorff convergence $(M_n, m_n) \to (M, m)$ with ε_n-approximations f_n, there exists a sequence $\{v_k\}_{k\in\mathbb{N}} \subset C_0(M)$ with $v_k \to u$ in $L_2(M, m)$ such that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \int_{M_n} |v_k \circ f_n - u_n|^2 \, dm_n = 0 \tag{3.3.6}$$

and therefore

$$\begin{aligned} &\lim_{k\to\infty} \limsup_{n\to\infty} \int_{\widehat{M_n}\cap M_n^{tube}} \left|v_k \circ \widehat{f}_n - u_n\right|^2 \, dm_n \\ &= \lim_{k\to\infty} \limsup_{n\to\infty} \int_{\widehat{M_n}\cap M_n^{tube}} |v_k \circ f_n - u_n|^2 \, dm_n = 0. \end{aligned}$$

We have

$$\begin{aligned} \int_{\widehat{M_n}\setminus M_n^{tube}} \left|v_k \circ \widehat{f}_n \right. &- \left. u_n\right|^2 dm_n \\ &\leq 2\int_{\widehat{M_n}\setminus M_n^{tube}} \left|v_k \circ \widehat{f}_n\right|^2 \, dm_n + 2\int_{\widehat{M_n}\setminus M_n^{tube}} |u_n|^2 \, dm_n \\ &\leq 2\left\|v_k\right\|_{L^\infty} m_n(\widehat{M_n} \setminus M_n^{tube}) + 2\int_{\widehat{M_n}\setminus M_n^{tube}} |u_n|^2 \, dm_n, \end{aligned}$$

and from (3.3.6) we deduce

$$\begin{aligned}
\limsup_{n\to\infty}\int_{\widehat{M_n}\setminus M_n^{tube}}|u_n|^2\ dm_n &\le 2\lim_{k\to\infty}\limsup_{n\to\infty}\int_{\widehat{M_n}\setminus M_n^{tube}}|v_k\circ f_n-u_n|^2\ dm_n\\
&+ 2\lim_{k\to\infty}\limsup_{n\to\infty}\int_{\widehat{M_n}\setminus M_n^{tube}}|v_k\circ f_n|^2\ dm_n\\
&\le 2\lim_{k\to\infty}\limsup_{n\to\infty}\|v_k\|_{L^\infty}m_n(\widehat{M_n}\setminus M_n^{tube})=0
\end{aligned}$$

and consequently

$$\lim_{k\to\infty}\limsup_{n\to\infty}\int_{\widehat{M_n}\cap M_n^{tube}}\left|v_k\circ\widehat{f}_n-u_n\right|^2\ dm_n=0.$$

Since $\{v_k|_{e_1\cup e_2\cup\{O\}}\}_k\subset C_0(e_1\cup e_2\cup\{O\})$ we conclude that $u_n|_{\widehat{M_n}}\to u|_{e_1\cup e_2\cup\{O\}}$ strongly for the measured Gromov-Hausdorff convergence

$$(\widehat{M_n},m_n|_{\widehat{M_n}})\to\left(e_1\cup e_2\cup\{O\},m|_{e_1\cup e_2\cup\{O\}}\right)$$

with ε_n-approximations $\widehat{f}_n$, which is equivalent to the strong convergence $u_n\circ\Psi_n\to u\circ\Psi$ for the measured Gromov-Hausdorff convergence $(J_n,m'_n)\to(J,m')$ with ε_n-approximations φ_n. Because $u_n\in H^1(M_n)$ we get $u_n\circ\Psi_n\in H^1(J_n)$ and from the Γ-convergence that we proved for cylindrical tubes within the subsection 3.2.1 we deduce $u\circ\Psi\in H^1(J)\subset C(J)$ and thus $u\in C(e_1\cup e_2\cup\{O\})$. In a similar way we prove $u\in C(e_i\cup e_j\cup\{O\})$ for i, $j=1,...,N$, $i\neq j$, therefore $u\in C(M)$, which ends the proof of (3.3.4).

In order to prove the second property (3.1.9) from the definition of the Γ-convergence we consider $u\in L_2(M,m)$ and we define $u_n:=u\circ f_n$, $n\in\mathbb{N}$. Obviously $u_n\to u$.

Since our ε_n-approximations $f_n:M_n\to M$ may be chosen Lipschitz, they satisfy the following two properties:

1. $u\in C(M)$ with $u|_{I_i}\in H^1(I_i)\ \forall i\ \Leftrightarrow\ u\circ f_n\in H^1(M_n)\ \forall n$ (the implication "$\Leftarrow$" was proved above)
2. $|\nabla(u\circ f_n)|^2\le k(u')^2\circ f_n\ \forall n$ with $k>0$ constant.

Because one inequality from (3.1.9) was proved above, it remains to show

$$\mathcal{E}(u)\ge\limsup_{n\to\infty}\mathcal{E}^n(u_n)\tag{3.3.7}$$

and we may suppose that $\mathcal{E}(u)<\infty\Leftrightarrow u\in C(M)$ with $u|_{I_i}\in H^1(I_i)$, $\forall i$. According to our assumption we have $u\circ f_n\in H^1(M_n)\ \forall n$. We know that $\mathcal{E}^n_{M_n^{tube}}(u\circ f_n)=\mathcal{E}_{M\cap M_n^{tube}}(u)$ from the cylindrical case and then

$$\limsup_{n\to\infty}(\mathcal{E}^n(u_n)-\mathcal{E}(u))=\limsup_{n\to\infty}\left(\mathcal{E}^n_{M_n\setminus M_n^{tube}}(u_n)-\sum_{i=1}^N\int_{e_i\setminus M_n^{tube}}|u'|^2\,dm\right)$$

$$
\begin{aligned}
&= \limsup_{n\to\infty}\left(\int_{M_n\setminus M_n^{tube}} \left|\nabla\left(u\circ f_n\right)\right|^2 dm_n - \sum_{i=1}^{N}\int_{e_i\setminus M_n^{tube}} \left|u'\right|^2 dm\right)\\
&\le k\limsup_{n\to\infty}\left(\int_{M_n\setminus M_n^{tube}} \left|u'\right|^2\circ f_n dm_n - \sum_{i=1}^{N}\int_{e_i\setminus M_n^{tube}} \left|u'\right|^2 dm\right)\\
&= \limsup_{n\to\infty} k\left(\int_{M_n} \left|u'\right|^2\circ f_n dm_n - \int_{M_n^{tube}} \left|u'\right|^2\circ f_n dm_n \sum_{i=1}^{N}\int_{e_i\setminus M_n^{tube}} \left|u'\right|^2 dm\right)\\
&= \limsup_{n\to\infty} k\left(\int_{M_n} \left|u'\right|^2\circ f_n dm_n - \int_{M\cap M_n^{tube}} \left|u'\right|^2 dm - \sum_{i=1}^{N}\int_{e_i\setminus M_n^{tube}} \left|u'\right|^2 dm\right) = 0.
\end{aligned}
$$

(ii) Let us suppose that $\{u_n\}_{n\in\mathbb{N}}$ is a sequence with $u_n \in L_2\left(M_n, m_n\right)$ such that $\sup_{n\in\mathbb{N}}(\mathcal{E}^n\left(u_n\right)+\left\|u_n\right\|^2_{L_2(M_n,m_n)}) < \infty$. We have to find a subsequence $\{u_{n_k}\}_{k\in N}$ strongly convergent. We denote by $\widetilde{M}_{n,1} = f_n^{-1}\left(e_1\right)$ which is an open subset of M_n and we consider the maps $\Psi_n : J_n \to \widetilde{M}_{n,1}$ with J_n the open cylindrical tube of radius $1/n$ around the segment J, Ψ_n bijections with $\Psi_n \in C^1\left(J_n\right)$, $\Psi_n^{-1} \in C^1(\widetilde{M}_{n,1})$, Ψ_n bi-Lipschitz maps. The segment e_1 is identified with J by the map $\Psi : J \to e_1$. We have proved already the asymptotic compactness for cylindrical tubes. According to Lemma 3.3.3 stated below we have

$$
\sup_{n\in N}\left(\mathcal{E}_J^n\left(u_n\circ\Psi_n\right) + \left\|u_n\circ\Psi_n\right\|_{L_2(J_n,m_n)}\right) < \infty
$$

and therefore there exists a subsequence $\{u_{n_k}\circ\Psi_{n_k}\}_{k\in\mathbb{N}}$ strongly convergent to a $v_1 \in H^1\left(J\right)$ or equivalently $\left\{u_{n_k}|_{\widetilde{M}_{n,1}}\right\}_{k\in\mathbb{N}}$ is strongly convergent to $v_1\circ\Psi^{-1} \in H^1\left(e_1\right)$. We replace now the initial sequence $\{u_n\}_{n\in\mathbb{N}}$ by $\{u_{n_k}\}_{k\in\mathbb{N}}$ for the simplicity of the notation and we repeat the procedure for the edges e_2 and e_3, ..., e_{N-1} and e_N, and we obtain a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $\left\{u_{n_k}|_{\widetilde{M}_{n,i}}\right\}_{k\in\mathbb{N}}$ is strongly convergent to $u^i := v_i\circ\Psi^{-1} \in H^1\left(e_i\right)$, $i = 1, ..., N$. We define $u = u^i$ on e_i, $i = 1, ..., N$. For each $i = 1, ..., N$ there exists a sequence $\{v_j^i\}_{j\in\mathbb{N}} \subset C_0\left(e_i\right)$ L_2-convergent to u^i such that

$$
\lim_{j\to\infty}\limsup_{k\to\infty}\int_{\widetilde{M}_{n,i}} \left|v_j^i\circ f_{n_k} - u_{n_k}|_{\widetilde{M}_{n,i}}\right|^2\ dm_{n_k} = 0
$$

Then the functions $v_j := v_j^i$ on e_i, $v_j\left(O\right) = 0$ belong to $C_0\left(M\right)$, the sequence $\{v_j\}_{j\in\mathbb{N}}$ is L_2-convergent to u and

$$
\lim_{j\to\infty}\limsup_{k\to\infty}\int_{M_n} \left|v_j\circ f_{n_k} - u_{n_k}\right|^2\ dm_{n_k} = 0,
$$

which proves the strong convergence of $\{u_{n_k}\}_{k\in\mathbb{N}}$ to u. □

Lemma 3.3.3. *Let Q_n, $n \in N$ be open subsets of R^3 such that there exist the maps $\Psi_n : J_n \to Q_n$ with J_n the open cylindrical tube of radius $\frac{1}{n}$ around the segment J ($J_n = J \times B_2(0, 1/n)$), Ψ_n bijections with $\Psi_n \in C^1(J_n)$, $\Psi_n^{-1} \in C^1(Q_n)$ and Ψ_n are bi-Lipschitz maps. Suppose that J_n and J are equipped with metrics and measures m_n and m like in Proposition 3.2.6, Q is either a segment or the union of two segments of R^3 and denote $\Psi : J \to Q$ the isometry between J and Q that preserves the singularity. We consider the maps $g_n : Q_n \to Q$, $g_n := \Psi \circ \varphi_n \circ \Psi_n^{-1}$ where $\varphi_n : J_n \to J$ are the projections that we used in subsection 3.2.1. If we denote $m'_n := m_n \circ \Psi_n$ and $m' := m \circ \Psi$ then $(Q_n, m_n) \to (Q, m)$ in the sense of measured Gromov-Haussdorf convergence with ε_n-approximations g_n. Moreover, $L_2(Q_n, m'_n) \ni u_n \to u \in L_2(Q, m')$ strongly if and only if $L_2(J_n, m_n) \ni u_n \circ \Psi_n \to u \circ \Psi \in L_2(J, m)$ strongly, $u_n \in H^1(Q_n)$ if and only if $u_n \circ \Psi_n \in H^1(J_n)$ and there exists $C > 0$ such that*

$$\frac{1}{C} \|u_n \circ \Psi_n\|_{L_2(J_n, m_n)} \leq \|u_n\|_{L_2(Q_n, m_n")} \leq C \|u_n \circ \Psi_n\|_{L_2(J_n, m_n)} \tag{3.3.8}$$

$$\frac{1}{C} \mathcal{E}_J^n(u_n \circ \Psi_n) \leq \mathcal{E}_Q^n(u_n) \leq C \mathcal{E}_J^n(u_n \circ \Psi_n) \tag{3.3.9}$$

Proof. We showed in Proposition 3.2.6 that $(J_n, m_n) \to (J, m)$ in the sense of the measured Gromov-Hausdorff convergence, and our hypothesis about the maps Ψ_n, $n \in \mathbb{N}$ and Ψ ensures us that (J_n, m_n) and (Q_n, m'_n), $n \in \mathbb{N}$, (J, m) and (Q, m') respectively have the same isomorphism classes. The fact that $u_n \in H^1(Q_n)$ if and only if $u_n \circ \Psi_n \in H^1(J_n)$ is well-known (see for instance Proposition IX.6 in [Bre92]). The proof of inequalities (3.3.8) and (3.3.9) consists in applying the formula of changing the variables for the integrals that appear and using the fact that the partial derivatives of Ψ_n and Ψ_n^{-1} are bounded, since Ψ_n are bi-Lipschitz maps. □

Corollary 3.3.4. *The sequence $\{\mathcal{E}^n\}_{n \in \mathbb{N}}$ compactly converges to $\mathcal{E}$.*

There exists a unique self-adjoint and non-negative operator L associated with $\mathcal{E}$, whose domain consists of those $u \in C(M)$ with $u|_{e_i} \in H^2(I_i)$, $\forall i$.

On an edge e_i the operator L is given by

$$Lu = -\frac{1}{\rho_i}(\rho_i u'_i)'. \tag{3.3.10}$$

Moreover, L satisfies the Kirchhoff boundary condition in the vertex O:

$$\sum_{i=1}^{N} \rho_i(O) u'_i(O) = 0, \tag{3.3.11}$$

where the derivative is taken on each edge in the direction away from the vertex O.

Indeed, from the condition

$$\mathcal{E}(u,u) = \langle Lu, u\rangle \text{ for any } u \text{ in the domain of } L$$

we derive

$$\sum_{i=1}^{N}\int_0^{l_i}(u_i')^2\rho_i dx = -\sum_{i=1}^{N}\int_0^{l_i}(\rho_i u_i')' u_i dx = -\sum_{i=1}^{N}(\rho_i u_i' u_i)|_0^{l_i} + \sum_{i=1}^{N}\int_0^{l_i}(u_i')^2\rho_i dx$$

for any $u \in C(M)$ with $u|_{e_i} \in H^2(I_i)$, $\forall i$. Since u is continuous in O we obtain the Kirchhoff boundary condition (3.3.11) in the vertex O, plus a Neumann boundary condition in the loose vertices $u'(l_i) = 0$ for each i.

From Theorem 3.1.10 and Corollary 3.3.4 the following result is straightforward.

Corollary 3.3.5. *For the corresponding strongly continuous contraction semigroups and the strongly continuous resolvents associated with $\mathcal{E}$ and $\mathcal{E}^n$ we have:*
(i) $R_\zeta^n \to R_\zeta$ compactly for some $\zeta < 0$;
(ii) $T_t^n \to T_t$ compactly for some $t > 0$.

Our approximating domains M_n don't necessarily have a smooth boundary, but they are at least Lipschitz, in the sense that locally, ∂M_n can be written as the (euclidian) graph of a Lipschitz function with M_n lying on one side of the graph. On such domains a Rellich compact embedding theorem still holds (see [Ros98]). Since the Rellich compact embedding theorem gives the compactness of the resolvent for bounded domains, from Theorem 3.1.10 we obtain also the convergence of spectra of the associated generators L_n:

Corollary 3.3.6. *The kth eigenvalue of L_n converges to the kth eigenvalue of L as $n \to \infty$ for any k.*

Remark 3.3.7. The convergence of spectra of the Neumann Laplacian on graph-like compact manifolds has been treated quite extensively in the paper [EP05]. They analyze there graph-like manifolds that around the edges behave like cylindrical neighborhoods with weights and three different cases of vertex-neighborhoods, that produce different operators in the limit. The limit operator on the graph depends on whether the vertex neighborhood decays (in volume) faster, slower, or at the same rate with the edge-neighborhood.

Our study considered open subsets of $\mathbb{R}^3$, a more divers class of edge-neighborhoods and a decay of the volume of the vertex-neighborhood that should be faster than the one of the edge-neighborhoods. Besides, the Kuwae-Shioya approach gives the convergence of the whole structure heat kernel-Dirichlet form-resolvent-semigroup on our open domain towards the one on the graph. The other two cases that [EP05] solved give in the limit some operators that are not defined on $L_2(M, m)$, but on a more general Hilbert space that contains $L_2(M, m)$ as a subspace. Therefore, we cannot expect that the Kuwae-Shioya theory in its actual form could handle those two cases too.

In fact, the convergence of spectra has been investigated intensively in the last years, also for boundary conditions other than Neumann. Mixed boundary conditions for the approximating sequence of manifolds have been considered in [Po05], [Gr07]. In [Po05] for instance the main result states the convergence of the spectra of a family of approximating open sets from $\mathbb{R}^2$ with small vertex neighborhoods and with a mixed boundary condition towards the spectrum of the Laplacian on the graph with Dirichlet boundary condition, which is actually a graph operator without coupling between edges. The paper [Po06] studies the approximations with non-compact manifolds and in the Neumann case gives, besides the convergence of spectra, the norm convergence of resolvents.

Bibliography

[Al51] A. D. Alexandrov (1951): A theorem on triangles in a metric space and some applications. *Trudy Math. Inst. Steklov* **38**, 5–23. (Russian; translated into German and combined with more material in [Al57])

[Al57] A. D. Alexandrov (1957): Über eine Verallgemeinerung der Riemannschen Geometrie.*Schr. Forschungsinst. Math. Berlin* **1**, 33–84.

[Am78] C.J. Amick (1978): Some remarks on Rellichs theorem and the Poincare inequality. *J. London Math. Soc. (2)* **18** no. 1, 81--93.

[BE85] D. Bakry, M. Émery (1985): Diffusions hypercontractives(French). [Hypercontractive diffusions], Séminaire de probabilités, XIX, 1983/1984, 177–206, *Lecture Notes in Math.*, 1123, Springer, Berlin.

[BG99] S. Bobkov, F. Götze (1999): Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal.* **163**, 1–28.

[Bo07] M. Bonnefont (2007): *A discrete version and stability of Brunn-Minkowski inequality*, preprint 2007.

[Bre92] H. Brezis (1992): *Analyse fonctionnelle*, Masson Paris.

[CE05] D. Cordero-Erausquin: private communication.

[DM93] G. Dal Maso (1993): *An Introduction to* Γ*-convergence*, Birkhäuser Boston Inc., Boston, MA.

[Du89] R.M. Dudley (1989): *Real analysis and probability.* The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA.

[EP05] P. Exner, O. Post (2005): Convergence of spectra of graph-like thin manifolds, *J. Geom. Phys.* **54** no. 1, 77–115.

[ES89] P. Exner, P. Šeba (1989): Bound states in curved quantum waveguides, *J. Math. Phys.* **30** no. 11, 2574–2580.

[Fo03] R. Forman (2003): Bochner's method for cell complexes and combinatorial Ricci curvature. *Discrete Comput. Geom.* **29** no. 3, 323–374.

[Fu87] K. Fukaya (1987): Collapsing of Riemannian manifolds and eigenvalues of Laplace operator, *Invent. Math.* **87**, 517–547.

[FOT94] M. Fukushima, Y. Oshima, M. Takeda (1994): *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter.

[Gr07] D. Grieser (2007): *Spectra of graph neighborhoods and scattering*, preprint 2007.

[Gro87] M. Gromov (1987): Hyperbolic groups. Essays in group theory, 75–263, *Math. Sci. Res. Inst. Publ.* 8, Springer, New York.

[Gro99] M. Gromov (1999): *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser Boston Inc., Boston, MA, Based on the 1981 French original [MR 85e:53051], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from French by Sean Michael Bates.

[Hi01] Y. Higuchi (2001): Combinatorial curvature for planar graphs. *J. Graph Theory* **38** no. 4, 220–229.

[Is90] M. Ishida (1990): *Pseudo-curvature of a graph*, lecture at "Workshop on topological graph theory", Yokohama national University.

[Kol05] A. Kolesnikov (2005): Convergence of Dirichlet forms with changing speed measures, *Forum Math.* **17**, 225–259.

[KoS99] V. Kostrykin, R. Schrader (1999): Kirchhoff's rule for quantum wires, *J. Phys. A* **32** no. 4, 595--630.

[KS03] K. Kuwae, T. Shioya (2003): Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry, *Comm. Anal. Geom.* **11** (4), 599–673.

[Ku02] P. Kuchment (2002): Graph models for waves in thin structures, *Waves in Random Media* **12**, R1–R24.

[Le01] M. Ledoux (2001): *The concentration of measure phenomenon.* Mathematical Surveys and Monographs **89**, American Mathematical Society.

[LV06] J. Lott, C. Villani (2006): Ricci curvature for metric-measure spaces via optimal transport, to appear in *Ann. of Math.*

[Ma97] K. Marton (1997): A measure concentration inequality for contracting markov chains. *Geom. Funct. Anal.* **6**, 556–571.

[Mos94] U. Mosco (1994): Composite media and asymptotic Dirichlet forms, *J. Funct. Anal.* **123**, 368–421.

[MR92] Z.M. Ma, M. Röckner (1992): *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*, Springer Universitext.

[Ne67] J. Nečas (1967): *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris.

[Po05] O. Post (2005): Branched quantum wave guides with Dirichlet boundary conditions: the decoupling case, *J. Phys. A* **38** no. 22, 4917–4931.

[Po06] O. Post (2006): Spectral convergence of quasi-one-dimensional spaces, *Ann. Henri Poincaré* **7** no. 5, 933–973.

[Ro03] J. Roe (2003): *Lectures on coarse geometry*, University Lecture Series, 31, American Mathematical Society, Providence.

[Ros98] M. Ross (1998): A Rellich-Kondrachov theorem for spiky domains, *Indiana Univ. Math. J.* **47** no. 4, 1497–1509.

[RS05] M.-K. von Renesse, K. T. Sturm (2005): Transport inequalities, gradient estimates, entropy and Ricci curvature, *Comm. Pure Appl. Math.* **68**, 923-940.

[St06a] K.T. Sturm (2006): On the geometry of metric measure spaces. I. *Acta Math.* **196** no. 1, 65–131.

[St06b] K.T. Sturm (2006): On the geometry of metric measure spaces. II, *Acta Math.* **196** no. 1, 133–177.

[SW04a] N. Sidorova, O.G. Smolyanov, H. v. Weizsäcker, O. Wittich (2004): The surface limit of Brownian motions in tubular neighborhoods of an embedded Riemannian manifold, *J. Funct. Anal.* **206** no. 2, 391–413.

[SW04b] N. Sidorova, O.G. Smolyanov, H. v. Weizsäcker, O. Wittich (2003): Brownian motion close to submanifolds of Riemannian manifolds, *Recent developments in stochastic analysis and related topics*, World Sci. Publ., Hackensack, NJ, 439–452.

[Ta96] M. Talagrand (1996): Transportation cost for Gaussian and other product measures, *Geom. Funct. Anal.* **6** no. 3, 587–600.

[Vi03] C. Villani (2003): *Topics in Mass Transportation.* Graduate Studies in Mathematics, American Mathematical Society.

[Vi08] C. Villani (2008): *Optimal transport, old and new.* Lecture Notes for the Saint-Flour 2005 course.

[Zhi98] V.V. Zhikov (1998): On weighted Sobolev spaces. (Russian) *Mat. Sb.* **189** (8), pp. 27–58; translation in *Sb. Math.* **189** (7-8) (1998), pp. 1139–1170.

Acknowledgements

The following is my heartfelt appreciation to those people, both past and present, who gave me spirit and encouragement to start and complete this thesis:

I would like to thank in the first place to my advisor Prof. Dr. Karl-Theodor Sturm for giving me the chance to perform my PhD in Bonn, for his scientific support, for constructive criticism, for his patience and advice.

I am grateful and indebted to Prof. Dr. Lucian Beznea, for all the help and for his guidance starting with the university years and lasting throughout the time I spent at my home institute in Romania.

My thanks go also to my former colleagues in the research team here in Bonn: Kathrin Bacher, Tom Christiansen, Atle Hahn, Martin Hesse, Ryad Husseini, Nicolas Juillet, Gustav Paulik, Robert Philipowski, Max ('Kostja') von Renesse, Ann-Kathrin Röwer, Hendrik Weber for interesting discussions and for a nice working environment.

Special thanks go to my family in Romania, to my son, to my husband and to my mother for their moral support and understanding through those difficult times when I was away, for their unconditional love.

Finally, I wish to thank all my friends whose names are not mentioned here, they all know who they are.

Participation in conferences and workshops

-International Conference on Complex Analysis and Related Topics "The IX^{th} Romanian-Finnish Seminar", August 27–31, 2001, Brasov, Romania.

-Summer school "Connections between Potential Theory, Geometry and Probability", August 27–September 6, 2001, Brasov, Romania.

-The 2^{nd} IMAR Workshop-Potential Theory, Bucharest, Romania, September 9–29, 2002.

-Summer School "Singular Phenomena and Scaling in Mathematical Models", Bonn, June 10–13, 2003.

-Potential Theory Conference, Bucharest, September 23–27, 2003.

-Miniconference: "Stochastic Models in Finance and Insurance", Bonn, November 6–7, 2003.

-Conference "Potential Theory and Related Topics", Hejnice, Czech Republic, September 26–October 2, 2004.

-Symposium on Stochastic Calculus with Applications in Geometry and Analysis, December 13-15, 2004, Bonn.

-International Conference on Complex Analysis and Related Topics, The Xth Romanian-Finnish Seminar, August 14-19, 2005, Cluj-Napoca, Romania.

-Symposium "Dirichlet Forms, Stochastic Analysis and Interacting Systems", November 21–26, 2005, Bielefeld/Bonn.

-Conference "Heat Kernels, Stochastic Processes and Functional Inequalities", November 27th–December 3rd, 2005, Oberwolfach.

-SFB 611 Winter School, February 13–17, 2006, Bonn.

-The German-Japanese Symposium "Stochastic Analysis and Applications", Kyoto University, Japan, September 11–15, 2006.

-The $VIth$ Congress of the Romanian Mathematicians, Bucharest, Romania, June 28–July 4, 2007.

-Conference "Potential Theory and Stochastics", Albac, Romania, September 4–8, 2007.

Scholarships

-October 1st, 2003–July 31st, 2006, DAAD scholarship, Institute of Applied Mathematics, University of Bonn.

Published papers

A.I. Bonciocat, On Beurling-Deny formula for quasi-regular Dirichlet forms, *Math. Rep.* **4**(54), no. 2, 143–146 (2003).

Bonn, 28/01/2008